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NOTES ON MAGNETOHYDRODYNAMICS - PART I

by
R. P. Kanwal

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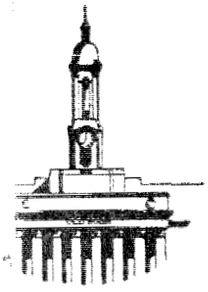

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Preface

This set of notes is based on a sequence of lectures which the author has given at this University for several years. The topics included are the equations of magnetohydrodynamics, general properties, boundary value problems, sound waves and shock waves. In a subsequent set of notes we plan to include simple waves, drag problems in non-viscous hydromagnetic flows, boundary layer, stability and turbulence. There will also be a chapter on selected topics such as superposable flows.

The subject of magnetohydrodynamics has aroused considerable interest and has produced a mountain of technical papers. My interest in the subject has been in the direction of investigating how the electrical conductivity of the fluid has modified the various results of classical hydrodynamics.

It is the present intent of the author to publish revisions of these and forthcoming notes in the form of a book. As such, it is hoped that many readers will send in their comments and suggestions for improvement.

Chapter I

Equations of Motion

A conducting fluid moving in a magnetic field induces an electric field and thereby electric currents are generated. The interaction of the magnetic field and the electric currents produces a mechanical force which modifies the motion. Conversely, the flow of an electrically conducting fluid modifies the magnetic field. When the fluid is regarded as a continuous medium, its motion is analyzed with the help of Navier-Stokes equations. The mechanical force due to the electromagnetic origin introduces new terms in these equations. This necessitates the study of electromagnetic fields and we need the Maxwell equations which govern such a field. We then have a determinate set of differential equations which are the basis of magnetohydrodynamics.

1. Navier-Stokes Equations

Let ρ and $\underline{v}$ denote respectively the density and the velocity field, then the equation of continuity is

$$\frac{d\rho}{dt} + \rho \operatorname{div} \underline{v} = 0, \tag{1}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \operatorname{grad} .$$

The equations of motion are

$$\rho \frac{d\underline{v}}{dt} = \rho f + \operatorname{div} \underline{\underline{T}} , \tag{2}$$

where $\underline{f}$ is the external force. $\underline{\underline{T}}$ is the stress tensor and shall be taken equal to $(-\rho + \lambda \operatorname{div} \underline{v})\, \underline{\underline{I}} + 2\mu \underline{\underline{D}}$. $\underline{\underline{I}}$ stands for δij and

$\underline{\underline{D}} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ is the deformation tensor. The quantities λ and μ are the coefficients of viscosity. If we take λ and μ to be constants, the equations (2) become

$$\rho \frac{d\underline{v}}{dt} = \rho \underline{f} - \text{grad } \rho + (\lambda + \mu) \text{ grad div } \underline{v} + \mu \nabla^2 \underline{v}. \qquad (3)$$

The equation of energy is

$$\rho \frac{dE}{dt} = \underline{\underline{T}} : \underline{\underline{D}} - \text{div } \underline{q}\ , \qquad (4)$$

where E is the interval energy of the fluid and $\underline{q}$ is the heat flux vector. If we assume that E is a thermodynamic state variable satisfying the relation with temperature T as

$$T\,dS = dE + p\,d\left(\frac{1}{\rho}\right),$$

the equation (4) becomes

$$\rho T \frac{dS}{dt} = \Phi - \text{div } \underline{q}\ , \qquad (5)$$

The function Φ is the dissipation function equal to $\underline{\underline{V}} : \underline{\underline{D}}$, and $\underline{\underline{V}} = (\lambda \text{ div } \underline{v})\,\underline{\underline{I}} + \mu\,\underline{\underline{D}}$, is the viscosity tensor. The heat flux vector $\underline{q}$ is related to temperature by the relation

$$\underline{q} = -\chi \text{ grad } T, \qquad (6)$$

the quantity χ is the coefficient of heat conductivity.

The appropriate equations of state relating the thermodynamic variables make the above system of equations a determinate system.

2. Maxwell Equations

As pointed out in the first paragraph, these equations have to be augmented in the case of conducting fluids. Due to the motion of a conducting fluid in a magnetic field, an electric field is induced. The electric field gives rise to electric currents. The law which relates these quantities is given by Ohm's law:

$$\underline{J} - Q\,\underline{v} = \sigma(\underline{E} + \underline{v} \times \underline{B}). \tag{7}$$

In this equation $\underline{J}$ is the electric current density, Q is the charge density, $\underline{E}$ is the electric field, $\underline{B}$ is the density of magnetic flux and σ is the electrical conductivity of the fluid. The quantity σ is assumed constant.

These four new quantities $\underline{J}$, Q, $\underline{E}$ and $\underline{B}$ enter our previous equations through the body force $\underline{f}$. The equation of energy is also modified due to an additional term i.e., Joule's heat. In this process we have introduced ten new quantities while we have added only three equations (7). Additional equations are provided by Maxwell's equations (in the M.K.S. system of units)

$$\frac{\partial Q}{\partial t} + \operatorname{div}\underline{J} = 0, \tag{8}$$

$$\operatorname{curl}\underline{E} + \frac{\partial \underline{B}}{\partial t} = 0, \tag{9}$$

$$\operatorname{div}\underline{B} = 0, \tag{10}$$

$$\operatorname{div}\underline{D} = Q, \tag{11}$$

$$\operatorname{curl}\underline{H} - \frac{\partial \underline{D}}{\partial t} = \underline{J}, \tag{12}$$

$$\underline{D} = \epsilon\,\underline{E}, \qquad \underline{B} = \mu_e\,\underline{H}, \tag{13}$$

Where μ_e is the magnetic permubility, assumed constant, $\underline{H}$ is the magnetic field intensity and $\underline{D}$ is the electric displacement. Let us observe that the equation (9) implies that $\frac{\partial}{\partial t}$ div $\underline{B}$ = 0. If we adopt the physically reasonable assumption that div $\underline{B}$ = 0, at any one time at a point then it will always be zero at that point. Therefore, the relation (10) is a consequence of the relation (9).

3. Equations of Magnetohydrodynamics

We now have all the necessary equations. However, magnetohydrodynamics is studied for a special case of the above set of equations. It is assumed that the displacement current $\frac{\partial \underline{D}}{\partial t}$ can be neglected except for a high frequency oscillation. When the displacement current is neglected, although the electrostatic effects of charge accumulations Q may be important, their effects on current flow must, for consistency, also be neglected. We are, thus, left with the system:

$$\text{curl } \underline{E} + \mu_e \frac{\partial \underline{H}}{\partial t} = 0 , \qquad \text{div } \underline{H} = 0, \tag{14}$$

$$\underline{J} = \text{curl } \underline{H}, \tag{15}$$

and

$$\underline{J} = \sigma (\underline{E} + \mu_e \underline{v} \times \underline{H}). \tag{16}$$

A glance over the above system of equations, would convince the reader that if $\underline{v}$ and $\underline{H}$ are known then $\underline{E}$ and $\underline{J}$ can be found by (15) and (16). We, therefore, derive an equation for $\underline{H}$ and $\underline{v}$ from the above set of equations. It is readily done by taking the curl of the equation (16) and substituting from (14) and (15). In fact the result is

$$\frac{\partial \underline{H}}{\partial t} = \text{curl}\,(\underline{v} \times \underline{H}) + \eta\, \nabla^2 \underline{H}, \tag{17}$$

where $\eta = \frac{1}{\mu_e \sigma}$ is known as the magnetic diffusivity. We add the mechanical force $\mu_e \underline{J} \times \underline{H}$, known as the Lorentz force on the right side of the equations of motion (2) or (3), and add the Joule's heat $\frac{J^2}{\sigma}$ on the right hand side of the energy equation (4) or (5). If there are no other external forces, we shall omit the quantity $\underline{f}$ from the equations of motion.

For future reference let us collect the required equations of magnetohydrodynamics together:

$$\text{div}\,\underline{H} = 0, \quad \underline{J} = \text{curl}\,\underline{H}, \tag{18}$$

$$\frac{\partial \underline{H}}{\partial t} = \text{curl}\,(\underline{v} \times \underline{H}) + \eta\, \nabla^2 \underline{H}, \tag{19}$$

$$\frac{d\rho}{dt} + \rho\, \text{div}\,\underline{v} = 0, \tag{20}$$

$$\rho \frac{d\underline{v}}{dt} = -\,\text{grad}\, p + (\lambda + \mu)\,\text{grad div}\,\underline{v} + \mu \nabla^2 \underline{v} + \mu_e \underline{J} \times \underline{H}, \tag{21}$$

$$\rho \frac{dS}{dt} = \Phi - \text{div}\,\underline{q} + \frac{J^2}{\sigma}, \tag{22}$$

and, of course, the thermodynamic relations.

4. Jump Relations across Transition Surfaces

In the next chapters we shall need the jump relations over surfaces across which the fluid flow and magnetic field quantities are discontinuous. Suppose a surface of discontinuity is propagating in the magnetized fluid with outward normal velocity G. Following is the

simple rule to write down the jump conditions.

Write the dynamical equations in the form

$$\frac{\partial F}{\partial t} + \nabla F = 0, \tag{23}$$

then replace

$$\frac{\partial}{\partial t} \text{ by } -G [\ ], \tag{24}$$

and

$$\nabla \text{ by } n [\ ], \tag{25}$$

where the bracket $[F]$ denotes the jump $(F_2 - F_1)$ of the required function F across the surface of the discontinuity, and $\underline{n}$ stands for the unit normal to this surface. It follows that

$$(\nabla .) = \text{div } F \text{ is replaced by } \underline{n} . [F], \tag{26}$$

and

$$(\nabla \times) = \text{curl } F \text{ is replaced by } \underline{n} \times [F]. \tag{27}$$

If we use the vector relations such as

$$\text{curl curl } F = \text{grad div} - \text{div grad},$$

and

$$\underline{J} \times \underline{H} = -\text{grad } \frac{H^2}{2} + \text{div } (\underline{H}\,\underline{H}),$$

a simple manipulation puts the equations of magnetohydrodynamics in the form

$$\text{div } H = 0, \tag{28}$$

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho v) = 0 \tag{29}$$

$$\frac{\partial}{\partial t} (\rho \underline{v}) + \text{div} \left\{ \rho \underline{v}\,\underline{v} + (\rho + \mu_e \frac{H^2}{2}) \underline{\underline{I}} - \underline{\underline{V}} - \mu_e \underline{H}\,\underline{H} \right\} = 0 \tag{30}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho E + \mu_e \frac{H^2}{2}\right) +$$

$$+ \operatorname{div}\left\{ \rho \underline{v} \left(\frac{1}{2} v^2 + \frac{p}{\rho} + E\right) + \mu_e \underline{H} \times (\underline{v} \times \underline{H}) - \right.$$

$$\left. - \frac{1}{\sigma} \underline{H} \times \operatorname{curl} \underline{H} - \underline{v} \cdot \underline{\underline{V}} + \underline{q} \right\} = 0 \qquad (31)$$

and

$$\frac{\partial \underline{H}}{\partial t} - \operatorname{curl} \left\{ (\underline{v} \times \underline{H}) - \eta \operatorname{curl} \underline{H} \right\} = 0 \qquad (32)$$

The quantity $\left(\frac{1}{2} \rho v^2 + \rho E + \mu_e \frac{H^2}{2}\right)$ is the total energy density.

By the above rule the jump conditions across a transition surface become

$$[H_n] = 0, \qquad (33)$$

$$- G [(\rho \underline{v})] + \left[\rho v_n \underline{v} + \left(p + \mu_e \frac{H^2}{2}\right) \underline{n} \right.$$

$$\left. - \underline{n} \cdot \underline{\underline{V}} - \mu_e H_n \underline{H} \right] = 0, \qquad (35)$$

$$- G \left(\frac{1}{2} \rho v^2 + \rho E + \mu_e H^2\right) + n \cdot \left[\rho \underline{v} \left(\frac{1}{2} \rho v^2 + \frac{p}{\rho} + E\right) + \right.$$

$$\left. + \mu_e \underline{H} \times (\underline{v} + \underline{H}) - \frac{1}{\sigma} \underline{H} \times \operatorname{curl} \underline{H} - \underline{v} \cdot \underline{\underline{V}} + \underline{q} \right] = 0$$

(36)

5. Dimensionless Parameters, Dynamic Similarity

Two motions, $(\underline{v}, \underline{H}, \rho, p)$ and $(\underline{v}', \underline{H}', \rho', p')$ are said to be dynamically similar if they are related by the equations

$$\underline{v} = U \underline{v}', \quad \underline{H} = h \underline{H}', \quad \rho = R \rho'$$

$$p = P p', \quad x = D x', \quad t = \tau t', \qquad (37)$$

where the quantities U, h, R, P, D, and τ are called the similarity constants. We shall show that these constants must be related in definite ways. That is to say that certain parameters must be the same for each flow.

Among these are many known parameters of non-conducting fluids. Most well-known are the Reynolds number, Prandtl number and Mach number. All the common parameters of the two dynamically similar flows of non-conducting fluids will remain common for the electrically conducting fluids as well. In the following analysis we investigate some new numbers which arise because of the conductivity of the fluid.

Making the substitutions (37) in the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \underline{v}) = 0,$$

we get

$$\frac{R}{T} \frac{\partial \rho'}{\partial t'} + \frac{RU}{D} \operatorname{div}' (\rho' \underline{v}') = 0. \tag{38}$$

Since the primed flow is also a solution of the equation of continuity, this implies that

$$\tau = D/U \tag{39}$$

Let us now take the field equation

$$\frac{\partial \underline{H}}{\partial t} + (\underline{v} \cdot \operatorname{grad}) \underline{H} - (\underline{H} \cdot \operatorname{grad}) \underline{v} + \underline{H} \operatorname{div} \underline{v} - \eta \nabla^2 \underline{H} = 0$$

and make the substitutions (37) and (39). After a slight simplification we obtain

$$\frac{\partial H'}{\partial t} + \underline{v}' \cdot \text{grad}' \underline{H}' - \underline{H}' \text{grad}' \underline{v}' + H' \text{div}' \underline{v}'$$

$$- \frac{\eta}{DU} \nabla'^2 \underline{H}' = 0. \tag{40}$$

For this equation to be equivalent for the corresponding field equation in primed quantities we should have

$$DU = \eta/\eta' . \tag{41}$$

From (41) it follows that the two flows must have the same parameter $L\, v/\eta$ at the corresponding points. This number is known as the magnetic Reynolds number. The quantity L is the representative length while $v^2 = \underline{v} \cdot \underline{v}$.

Similarly the equations of motion

$$\rho \frac{d\underline{v}}{dt} = - \text{grad } p + (\lambda \text{ grad div } v + 2\mu \text{ div } \underline{\underline{D}}) + \mu_e \underline{J} \times \underline{H},$$

give, after the above mentioned substitutions,

$$\rho' \frac{d\underline{v}}{dt} = - \frac{P}{RU^2} \text{grad}' p' + \frac{1}{RUD} (\lambda \text{ grad}' \text{div}' \underline{v}'$$

$$+ 2\mu \text{ div}' \underline{\underline{D}}') + \frac{h^2}{RU^2} \mu_e \underline{J}' \times \underline{H}' \tag{42}$$

This equation provides us with the relations

$$P = R U^2 \tag{43}$$

$$RUD = \lambda/\lambda' = \mu/\mu' \tag{44}$$

and

$$\frac{RU^2}{h^2} = \mu_e/\mu_e' \tag{45}$$

The relations (44) give the classical result, viz., the ratio λ/μ and the local Reynolds number $L\,v/\nu$ must be the same for the two flows.

The relation (45) states that the number $\mu_e H^2/\rho v^2$ must be the same for two flows at corresponding points. Here $H^2 = \underline{H} \cdot \underline{H}$. Since $\mu_e H^2/\rho = a^2$ is the square of the Alfven speed, the square root of this number is called the magnetic Mach number. It is also the ratio of the magnetic pressure to the dynamic pressure.

From relations (41) and (44) we get the result that the ratio of viscous and magnetic diffusivities, i.e., ν/η must be the same for the two fluids.

Finally we attend to the energy equation

$$\rho \frac{dE}{dt} + p \operatorname{div} \underline{v} = \lambda (\operatorname{div} \underline{v})^2 + 2\mu \underline{\underline{D}} : \underline{\underline{D}} + \chi \nabla^2 T + \frac{J^2}{\sigma} \quad .$$

We assume the gas law:

$$p = \mathcal{R} \rho T, \; E = c_v T, \tag{46}$$

where $\mathcal{R}$ is a positive constant and c_v is the coefficient of specific heat at constant volume.

From (7) and (15) we obtain

$$T = \frac{P}{\mathcal{R}} \frac{\mathcal{R}'}{R} \quad T' = T_o T', \tag{47}$$

which defines the similarity constants T_o. Now if we make the usual substitutions in the equation of energy as given above, there results

$$\frac{R T_o}{P} \quad c_v \rho' \frac{dT}{dt'} + \rho' \operatorname{div}' v' = \frac{U}{DP} (\lambda \theta'^2 + 2\mu \underline{\underline{D'}} : \underline{\underline{D'}}) \tag{48}$$

$$+ \frac{T_o}{DUP} \chi \nabla'^2 T' + \frac{h^2}{DUP} \frac{J'^2}{\sigma} \tag{48}$$

This gives rise to the relations

$$\frac{P}{RT_o} = \frac{c_v}{c'_v} \quad , \quad RUD \left(\frac{c_v}{c'_v} \right) = \frac{\chi}{\chi'}$$

and

$$\frac{h^2}{DUP} = \frac{\sigma}{\sigma'} \quad . \tag{49}$$

The relations $(49)_1$ and $(49)_2$ lead to Prandtl and Mach numbers. The relation $(49)_3$ is not new; in fact it is easily obtained from the relations (41) and (45).

Many other numbers can be formed from these relations. For example, in the subsequent chapters we shall need the number $\mu_e HL (\sigma/\mu)^{\frac{1}{2}}$, known as the Hartman number. To prove that this number should be the same for two dynamically similar flows, we observe that

$$\frac{\mu_e^2 H^2 L^2 \sigma \mu'}{\mu_e'^2 H'^2 L'^2 \sigma' \mu} = 1, \tag{50}$$

if we substitute from (37) and use the various relations given above.

6. Uniqueness of Magnetohydrodynamic Flows.

As in every branch of applied mathematics, it is natural to inquire whether the problem, as posed by the system of differential equations of this chapter, is well set. That is to say, whether the solutions of this system of equations actually exist, are unique and stable. So far only the problem of the uniqueness appears to have been discussed with some success and we give a brief account of

that analysis. In fact we prove that fluid motion in a bounded region $V = V(t)$, with boundary S, is uniquely determined by initial distributions of velocity, magnetic field, temperature and density together with the:

Boundary conditions: At all points of the boundary S, and at all instants of time, $\underline{v}$ and $\underline{H}$ are prescribed. Furthermore,

(a) At all points where $U = (\underline{v} \cdot \underline{n} - G) < 0$, the absolute temperature and the density are prescribed. The quantity G, as in previous sections, stands for the outward normal speed of the boundary S.

(b) At all points where $U > 0$, the absolute temperature is given.

(c) At all points where $U = 0$, one of the following conditions is assumed:

(c_1) The temperature is prescribed. (c_2) The normal heat flux $\underline{q} \cdot n$ is prescribed. (c_3) The normal heat flux is assumed to be proportional to the difference between the temperature T of the fluid and a given wall temperature T_o, i.e., $\underline{q} \cdot \underline{n} = k(T - T_o)$, $k > 0$.

It is assumed, in the following, that $\mu > 0$, $\mu_e > 0$, $\sigma > 0$ and $3\lambda + 2\mu > 0$ (the case $3\lambda + 2\mu = 0$, can be similarly proved). To prove our result we let $(\underline{v}, \underline{H}, \rho, T)$ and $(\tilde{\underline{v}}, \tilde{\underline{H}}, \tilde{\rho}, \tilde{T})$ be two solutions of the problem under discussion. A tilde over a quantity will mean that the quantity is to be evaluated for the flow and field $(\tilde{\underline{v}}, \tilde{\underline{H}}, \tilde{\rho}, \tilde{T})$. Also set $F' = \tilde{F} - F$. In particular $\tilde{\underline{v}} - \underline{v} = \underline{v}'$, etc. The following two simple identities will be used frequently in this section:

$$\widetilde{AB} - AB = AB' + \widetilde{\widetilde{A'B'}} \quad , \tag{51}$$

$$\frac{\widetilde{dF}}{dt} - \frac{dF}{dt} = \frac{dF'}{dt} + \underline{v}' \cdot \operatorname{grad} \widetilde{F}. \tag{52}$$

We shall also use the transport theorem in the following extended form

$$\frac{d}{dt} \int_V \rho F \, dV = \int_v \rho \frac{dF}{dt} \, dV - \oint_S \rho U F \, dS. \tag{53}$$

Since both flows satisfy the equations of magnetohydrodynamics we obtain by subtraction and by using identities (51) and (52):

$$\frac{d\rho'}{dt} = - v' \cdot \operatorname{grad} \rho - \rho \operatorname{div} v' - \rho' \operatorname{div} v, \tag{54}$$

$$\frac{dH'}{dt} = - \underline{v}' \cdot \operatorname{grad} \underline{H} - \underline{H} \operatorname{div} \underline{v}' + \underline{H} \cdot \operatorname{grad} \underline{v}'$$

$$+ \underline{H}' \cdot \operatorname{grad} \underline{v} + \eta \nabla^2 \underline{H}, \tag{55}$$

$$\frac{dv'}{dt} = \rho \underline{v}' \cdot \operatorname{grad} \widetilde{v} - \rho' \frac{\widetilde{dv}}{dt} + \rho' \underline{f} + \operatorname{div} \underline{\underline{T}}'. \tag{56}$$

$$\frac{dT'}{dt} = - \rho' \underline{v}' \cdot \operatorname{grad} T - \rho' \frac{\widetilde{dT}}{dk} + \psi' - \operatorname{div} q'$$

$$+ \frac{1}{\sigma} (\operatorname{curl} \underline{H} \cdot \operatorname{curl} \underline{H}' + \operatorname{curl} \underline{H}' \cdot \operatorname{curl} \widetilde{\underline{H}}), \tag{57}$$

where we have put $c_v = 1$, without any loss of generality, and $\psi = \underline{\underline{T}} : \underline{\underline{D}}$. Now multiply (54) and (57) by ρ, ρ' and T' respectively; form the scalar product of (55) and (56) by $\rho \underline{H}'$ and $\underline{v}'$ respectively; rearrange the terms; use the transport theorem (53), the divergence theorem, the boundary conditions and write

$$I = \int_V \frac{1}{2} \rho H^2 \, dV \, , \; J = \int_V \frac{1}{2} \rho \rho'^2 \, dV, \quad K = \int_V \frac{1}{2} \rho v'^2 \, dV$$

$$L = \int_V \frac{1}{2} \rho \, T'^2 \, dV, \tag{58}$$

getting thereby

$$\frac{dI}{dt} = \int_V \Big[- \rho v' \cdot (\operatorname{grad} \underset{\sim}{H}) \cdot \underline{H}' - \rho \underline{H} \cdot \underline{H}' \operatorname{div} \underline{v}' -$$

$$- \rho H'^2 \operatorname{div} \underset{\sim}{v} + \rho \underline{H}' \cdot (\operatorname{grad} \underset{\sim}{v}) \cdot H' -$$

$$- (\underline{H}' \cdot \underline{v}') (\underline{H} \cdot \operatorname{grad} \rho) - \rho \underline{H} \cdot (\operatorname{grad} \underline{H}') \cdot \underline{v}' -$$

$$- \eta \operatorname{grad} \rho \cdot (\operatorname{grad} \underline{H}') \cdot \underline{H}' - \eta \rho \operatorname{grad} \underline{H}' : \operatorname{grad} \underline{H}' \Big] dV \tag{59}$$

$$\frac{dJ}{dT} = - \int_V (\rho \rho' \underline{v}' \cdot \operatorname{grad} \underset{\sim}{\rho} + \rho^2 \rho' \operatorname{div} \underline{v}' + \rho \rho'^2 \operatorname{div} \underset{\sim}{v}) \, dV, \tag{60}$$

$$\frac{dK}{dt} = - \int_V \Big[\rho' \underline{v}' \cdot (\underline{a} - \underline{f}) + \rho \underline{v}' \cdot \underline{\underline{D}} \cdot \underline{v}' + \underline{\underline{T}}' : \underline{\underline{D}}' +$$

$$\rho \underline{v}' \cdot (\operatorname{grad} \underline{H}') \cdot \underline{H} + \rho \underline{v}' \cdot (\operatorname{grad} \underline{H}) \cdot H' -$$

$$- \rho \underline{H} \cdot (\operatorname{grad} \underline{H}) \cdot \underline{v}' - \rho \underline{H}' \cdot (\operatorname{grad} \underset{\sim}{H}) \cdot \underline{v}' - \operatorname{div} (\underline{\underline{T}}' \cdot \underline{v}') \Big] dV \tag{61}$$

$$\frac{dL}{dt} = \int_V \Big[(T' \psi' + q' \cdot \operatorname{grad} T' - \rho' T' \frac{dT}{dt} - \rho T' \underline{v}' \cdot \operatorname{grad} T) +$$

$$\frac{1}{\sigma} (T' \operatorname{curl} \underline{H} \cdot \operatorname{curl} H' + T' \operatorname{curl} \underline{H}' \cdot \operatorname{curl} \underset{\sim}{H}) \Big] dV$$

$$- \oint_S \Big[T' q' \cdot n + \frac{1}{2} U \, T'^2 \Big] \, dS \tag{62}$$

Finally we use the simple identities:

$$2\,ab \leq a^2 + b^2 ,$$

$$2\,ab = 2\,\frac{a}{\sqrt{\varepsilon}}\; b\sqrt{\varepsilon} \leq \frac{1}{\varepsilon}\; a^2 + \varepsilon\, b^2 ,$$ (ε is an arbitrary positive number to be fixed later)

$$\int_V \left| \operatorname{curl} \underline{H'} \cdot \operatorname{curl} \underline{H'} \right| dt = \int_V \left| (\operatorname{grad} \underline{H} : \operatorname{grad} \underline{H}) \right| dV$$

Then the relations (59) to (62) yield the inequalities:

$$\frac{dI}{dt} \leq \int_V N(v'^2 + H'^2)\, dV + \int_V \varepsilon\, (\theta'^2 + \left| \operatorname{grad} H' \right|^2) -$$

$$\eta\, \rho \left| \operatorname{grad} H' \right|^2 dV, \qquad 0 \leq t \leq \tau, \tag{63}$$

$$\frac{dJ}{dt} \leq \int_V N\,(\rho'^2 + v'^2)\, dV + \int_V \varepsilon\, v'^2\, dV, \tag{64}$$

$$\frac{dK}{dt} \leq \int_V N(\rho'^2 + v'^2 + p'^2 + H'^2)\, dV + \int_V (\varepsilon\, \theta'^2 + \left| \operatorname{grad} \underline{H'} \right|^2 -$$

$$- \underline{\underline{V'}} : \underline{\underline{D'}})\, dV, \tag{65}$$

$$\frac{dL}{dt} \leq \int_V N(\rho'^2 + v'^2 + p'^2 + T'^2) + \int_V (\varepsilon\, \theta'^2 + \underline{\underline{D}} : \underline{\underline{D'}} +$$

$$+ \left| \operatorname{grad} H' \right|^2)\, dV, \tag{66}$$

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* This relation holds when boundary conditions are taken into account.

where θ = div $\underline{v}$ and N is the generic notation for an upper bound. This quantity N depends on ε and on the bounds for the magnitude of ($\underline{v}$, $\underline{H}$, ρ, T), ($\tilde{\underline{v}}$, $\tilde{\underline{H}}$, $\tilde{\rho}$, $\tilde{T}$), and their derivatives during the time interval $0 \leq t \leq \tau$. It will differ for every estimate, but it is possible to determine its size at any stage. Also, we observe from the relation (46) that

$$p'^2 \leq N (\rho'^2 + T'^2) \ .$$

The relations (63) and (66) yield the inequality

$$\frac{d}{dt} (I + J + K + L) \leq N (I + J + K + L) + \Big[\varepsilon(3 \theta'^2 + D' : D') - \underline{\underline{V'}} : \underline{\underline{D'}} + (3\varepsilon - \rho\eta) \Big| \text{grad } H' \Big|^2 \Big] \tag{67}$$

Now

$$3 \theta'^2 + \underline{\underline{D'}} : \underline{\underline{D'}} = \frac{1}{3} (10 \theta'^2 + \Delta'^2) ,$$

and

$$\underline{\underline{V'}} : \underline{\underline{D'}} = \frac{1}{3} \left\{ (3\lambda + 2\mu) \theta'^2 + 2\mu \Delta'^2 \right\} .$$

where

$$\Delta^2 = (d_1 - d_2)^2 + (d_2 - d_3)^2 + (d_3 - d_1)^2,$$

and d_1, d_2 and d_3 are the principal values of D. We have taken ρ to be positive in the closure of V. Thus, if we choose ε small enough, the relation (67) becomes

$$\frac{d}{dt} (I + J + K + L) \leq N(I + J + K + L), \ 0 \leq t \leq \tau, \tag{68}$$

This differential inequality has the solution

$$I + J + K + L \leq (I + J + K + L)_o \, e^{NT}, \quad 0 \leq t \leq \tau \tag{69}$$

Since the integrals I, J, K and L are zero when t = 0, it follows that they remain zero throughout the whole interval in question. This implies $\underline{v}' \equiv \underline{H}' \equiv \rho' \equiv T' \equiv 0$, and so the two flows and fields are identical at least until $t = \tau$. Finally, τ being arbitrary, the flows and fields are identical so long as they exist, so that the result is now proved.

Chapter II

General Properties

We observe that the general equations of magnetohydrodynamics are non-linear. This system of equations is more complicated than the fundamental equations of viscous compressible fluids, i.e., Navier Stokes equations whose properties are not completely known as yet. We would expect that we know even much less of the mathematical properties of the present system of equations. However, some fruitful investigations have been carried out. We shall discuss some of them in this chapter.

1. Maxwell Stresses

First of all we observe that the mechanical force arising from electromagnetic interaction can be written as

$$\underline{J} \times \underline{H} = e_{lmm}\, e^{mjk}\, H_{k,j}\, H_n = \delta^{jk}_{nl}\, H_{k,j}\, H_n,$$

$$= -\operatorname{grad} \frac{H^2}{2} + \operatorname{div} (\underline{H}\,\underline{H}), \tag{1}$$

where e_{ijk} are the permutation tensors and we have used the fact that $\underline{H}$ is solenoidal. The expression

$$(-\mu_e \frac{H^2}{2} \underline{\underline{I}} + \mu_e \underline{H}\,\underline{H}),$$

is called the Maxwell stress tensor, while $\frac{\mu_e H^2}{2}$ is called the magnetic pressure.

We can understand something about the geometry of the Maxwell stress tensor by introducing the unit directions $\underline{t}$, $\underline{n}$ and $\underline{b}$, respectively along the tangent, principal normal and binormal to the lines of force $\underline{H}$. If we write $\underline{H} = H\underline{t}$, the expression (1) takes the form

$$\underline{J} \times \underline{H} = - \left\{ \nabla - \underline{t} (\underline{t} \cdot \nabla) \right\} \frac{H^2}{2} + \frac{\underline{n}}{R} H^2 , \qquad (2)$$

where R is the radius of curvature of the field lines. Thus, there is no contribution of this force in the direction of the field lines. In the direction $\underline{n}$ of the principal normal, the gradient of $\frac{H^2}{2}$ is reduced by $\frac{H^2}{R}$ while in the binormal direction the contribution comes only from the magnetic pressure term.

2. The Frozen-in Field

When the fluid is a perfect conductor, the field equation (1.19) reduces to

$$\frac{\partial \underline{H}}{\partial t} - \text{curl} (\underline{v} \times \underline{H}) = o, \qquad (3)$$

Now

$$\text{curl} (\underline{v} \times \underline{H}) = (\underline{H} \cdot \text{grad}) \underline{v} - (\underline{v} \cdot \text{grad}) \underline{H} - \underline{H} \, \text{div} \, \underline{v}. \qquad (4)$$

By the help of the equation of continuity the quantity div $\underline{v}$ can be replaced in terms of the derivatives of the density ρ. A slight rearrangement of the equation (3) then provides us with the equation

$$\frac{d}{dt} \left(\frac{\underline{H}}{\rho} \right) = \left(\frac{\underline{H}}{\rho} \cdot \text{grad} \right) \underline{v}. \qquad (5)$$

There is a formal equivalence between (5) and the equation for the transport of vorticity in an inviscid fluid. We can, therefore, invoke the famous Kelvin's circulation theorem and observe that for perfectly conducting fluids the magnetic lines of force are frozen in the fluid.

This idea can be grasped more easily by invoking another result, viz.

$$\frac{d \, d\underline{x}}{dt} = (d\underline{x} \cdot \text{grad}) \underline{v}, \qquad (6)$$

where $d\underline{x}$ is an element of length of some material fluid line. Heuristically, this result can be proved in the following way. If $\underline{v}$ is the fluid velocity at one end of the element $d\underline{x}$, then the fluid velocity at the other end is $\underline{v} + (d\underline{x} \cdot \text{grad})\,\underline{v}$. Then, during a time interval dt, the length of $d\underline{x}$ has changed by $dt\,(d\underline{x} \cdot \text{grad})\,\underline{v}$ and the above result follows. From this equivalence of relations (5) and (6) it follows that, if these vectors $\underline{H}/\rho$ and $d\underline{x}$ are initially in the same direction, they will remain parallel and their lengths will remain in the same ratio. Therefore, if two infinitely close fluid particles are on the same line of force at any time, then they will always be on that line of force. Passing now from particles at an infinitesimal distance apart to those at any finite distance apart, the concept of frozen-in fields becomes rather clear.

3. Bernoulli's theorem for Steady Flow

The equations of motion for steady incompressible and inviscid fluids are

$$\rho\,\underline{v} \cdot \text{grad}\,\underline{v} = -\nabla\left(p + \frac{\mu_e H^2}{2}\right) + \mu_e\,(\underline{H} \cdot \text{grad})\,\underline{H}$$

Using the identity

$$\underline{v} \cdot \text{grad}\,\underline{v} = (\text{curl}\,\underline{v}) \times \underline{v} + \text{grad}\left(\frac{1}{2}\,v^2\right)$$

and the relation (2) of section (1), the above equation can be written as

$$\underline{v} \cdot \nabla\left(p + \frac{\mu_e H^2}{2} + \frac{1}{2}\,\rho v^2\right) = \underline{v} \cdot \underline{n}\,\frac{\mu_e H^2}{R} + \underline{v} \cdot \underline{t}\,\underline{t} \cdot \nabla\frac{\mu_e H^2}{2}). \tag{7}$$

4. The Law of Isorotation

We can get some information from the field equation alone under the circumstances that the field is only poloidal, i.e., $\underline{H} = (H_r, o, H_z)$

in cylindrical polar coordinates while the velocity field is $\underline{v} = (o, r\omega, o)$. The steady field equation for infinitely conducting fluid is

$$\text{curl}\,(\underline{v} \times \underline{H}) = 0, \qquad \text{div}\,\underline{H} = 0\,. \tag{8}$$

But $(\underline{v} \times \underline{H})$ in the present case is $(-r\omega H_z, o, r\omega H_r)$. If we define a stream function Ψ such that

$$H = (H_r, o, H_z) \equiv \left(\frac{1}{r}\frac{\partial \Psi}{\partial z}, o, \frac{1}{r}\frac{\partial \Psi}{\partial z}\right), \tag{9}$$

the equation $(8)_2$ is automatically satisfied while the equation $(8)_1$ gives the Jacobian

$$\frac{\partial(\omega, \Psi)}{(r, z)} = 0 \tag{10}$$

Equation (10) implies that $\omega = f(\Psi)$. Thus the angular velocity is constant over a surface generated by the rotation of a field line about the axis.

5. Parallel stationary flow

Another interesting solution for the motion of a steady and incompressible fluid is provided by putting

$$\underline{v} = \sqrt{\frac{\mu_e}{\rho}}\ \underline{H}. \tag{11}$$

The field equation $\text{curl}\,(\underline{v} \times \underline{H}) = 0$ is identically satisfied. The flow equation

$$(\text{curl}\,\underline{v}) \times \underline{v} = \frac{1}{\rho}\nabla\left(p + \frac{1}{2}\rho v^2\right) + \frac{\mu_e}{\rho}(\text{curl}\,\underline{H}) \times \underline{H}$$

becomes

$$\nabla\left(p + \frac{1}{2}\rho v^2\right) = 0. \tag{12}$$

Thus if

$$p + \frac{1}{2} \rho v^2 = \text{constant} \tag{13}$$

both the field and flow equations are satisfied.

6. Magnetohydrostatic problems

When $\underline{v} = 0$, (14)

$$\frac{\partial \underline{H}}{\partial t} = \eta \nabla^2 \underline{H}. \tag{15}$$

$$\text{grad } p = \rho \underline{g} + \mu_e (\text{curl } \underline{H}) \times \underline{H}. \tag{16}$$

When ρ = a constant, $\underline{g} = -\text{grad } \Omega$, the equation (16) becomes

$$\text{grad } (p + \rho\Omega) = \mu_e \underline{J} \times \underline{H}.$$

Thus both $\underline{J}$ and $\underline{H}$ lie on the surfaces $p + \rho\ \Omega = \text{const}$.

The equation (16) can also be written as (ρ = a constant):

$$-\text{grad } (p + \rho\ \Omega + \frac{1}{2} + \mu_e H^2) - \mu_e \text{ div } (\underline{H}\ \underline{H}) = 0, \tag{17}$$

which shows that the lines of magnetic force behave like elastic strings with tension $\mu_e H^2$ and acted upon by the total hydrostatic pressure $p + \rho\Omega + \frac{1}{2} \mu_e H^2$.

7. Force-free fields

There is a considerable interest in the case in which the magnetic field exerts no force on the material, i.e., $(\text{curl } \underline{H}) \times \underline{H} = 0$. Such a field is called force-free and arises in regions where magnetic field is everywhere parallel to the direction of current flow. In this case the hydrostatic equilibrium is the same as in the absence of a magnetic field, provided the equations (14) and (15) are satisfied for $\underline{H}$. The condition $(\text{curl } \underline{H}) \times \underline{H} = 0$, implies

$$\text{curl } \underline{H} = \alpha \underline{H}, \tag{18}$$

where α is a scalar function of position and time. If we take the divergence of (18), we get

$$\underline{H} \cdot \text{grad } \alpha = 0. \tag{19}$$

Furthermore, if we take the curl of (18) and use the field equations, there results

$$\frac{1}{\eta} \frac{\partial \underline{H}}{\partial t} = - \alpha^2 \underline{H} + (\underline{H} \times \text{grad } \alpha). \tag{20}$$

For the field to remain force-free, we must have

$$\underline{H} = \underline{H}_o \ f(t), \tag{21}$$

where H_o is the field at time t = 0 and f(t) is a function of the time only. Substitution in (18-20) implies that α is independent of t and $H_o \cdot \text{grad } \alpha = 0$, as well as $H_o \times \text{grad } \alpha = 0$. Thus α is a constant. Taking the curl of the equation (18) and using the above analysis yields

$$\nabla^2 \underline{H}_o + \alpha^2 \underline{H}_o = 0. \tag{22}$$

In the sequel we shall drop the zero subscript. Solutions of this equation can be derived from the scalar Helmholtz equation::

$$\nabla^2 \psi + \alpha^2 \psi = 0, \tag{23}$$

in the following manner.

Let '$\underline{a}$' be any constant unit vector. Then three independent solutions of equation (22) are

$$\begin{aligned} &\underline{L} = \text{grad } \psi, \qquad \underline{T} = \text{curl } (\underline{a} \psi), \\ &S = \frac{1}{\alpha} \text{ curl } \underline{T} = \frac{1}{\alpha} \text{ curl curl } (a \psi). \end{aligned} \tag{24}$$

Since $\underline{a}$ is a constant vector, a slight algebraic manipulation makes the following results obvious.

$$\underline{T} = \underline{L} \times \underline{a} = \frac{1}{\alpha} \operatorname{curl} \underline{S}. \tag{25}$$

$$\underline{L} \cdot \underline{T} = 0, \operatorname{curl} \underline{L} = 0, \operatorname{div} \underline{T} = 0,$$

$$\operatorname{div} \underline{S} = 0, \operatorname{div} \underline{L} = -\alpha^2 \psi. \tag{26}$$

The solenoidal vectors $\underline{T}$ and $\underline{S}$ are called toroidal and poloidal vector fields; respectively. Since $\underline{H}$ is a solenoidal field the solutions of interest involve the toroidal vector $\underline{T}$ and poloidal vector $\underline{S}$. Moreover, from the relations

$$\operatorname{curl} \underline{S} = \alpha \underline{T}, \text{ and } \operatorname{curl} \underline{T} = \alpha \underline{S}, \tag{27}$$

we have

$$\operatorname{curl} (\underline{T} + \underline{S}) = \alpha (\underline{T} + \underline{S}). \tag{28}$$

Thus the most general solution of the equation $\operatorname{curl} \underline{H} = \alpha \underline{H}$, is

$$\underline{H} = \underline{T} + \underline{S}.$$

Examples:

(1) Let the solution depend on a single line or coordinate x, the scalar Helmholtz equation then becomes

$$\frac{d^2\psi}{dx^2} + \alpha^2 \psi = 0, \tag{29}$$

whose solution may be taken as $\psi = A \cos \alpha x$, where A is a constant. Taking $\underline{a}$ ▫ (o, o, 1), we find

$$\underline{T} = \left(o, \frac{-d\psi}{dx}, o\right), \underline{S} = (o, o, \alpha \psi). \tag{30}$$

Thus

$$\underline{H} = H_o (o, \sin \alpha x, \cos \alpha x), \tag{31}$$

where H_o is a constant. This field is uniform in any plane x = constant, but whose direction varies for different planes.

(2) Axially symmetric Helmholtz equation for ψ is

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \alpha^2 \psi = 0 \qquad (32)$$

where r is the distance from the axis of symmetry. This is Bessel's equation of zero order and has the solution $J_o(\alpha r)$.

Taking $\underline{a}$ = (o, o, 1), the value of $\underline{H}$ becomes

$$\underline{H} = H_o\,(o,\ J_1(\alpha r),\ J_o(\alpha r)) \ , \qquad (33)$$

where H_o is a constant. The lines of force of this force-free field are circular helics drawn on coaxial cylinders, the pitch of helics diminishing to zero on receding from the axis, then increasing again and so on.

Finally, we prove that there is equipartition of energy between toroidal and poloidal components of any force-free magnetic field within a spherical boundary. Equation (30), when written in spherical polar corrdinates r, θ, φ, becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \varphi^2} + \alpha^2 \psi = 0. \qquad (34)$$

A general solution $\psi_n^{\,m}$ for this equation is given as

$$\psi_n^{\,m} = Z_n(\alpha r)\, P_n^{\,m}(\cos\theta)\, e^{im\phi} \ , \qquad (35)$$

where m and n are integers, $P_n^m(\cos\theta)$ is the associated Legendre polynomial and $Z_n(\alpha r)$ is the spherical Bessel function which satisfies the ordinary differential equation:

$$r^2 \frac{d^2R}{dr^2} + 2r \frac{dR}{dr} + \left\{ \alpha^2 + n(n+1) \right\} R = 0. \qquad (36)$$

From (24), it follows that in this case $\underline{L} = \text{grad}(\psi_n^m)$. The associated toroidal and poloidal fields may then be derived from relations (24) and (25), by using a suitable unit vector $\underline{a}$. The field tangential to the spherical surface will be given by choosing a radial unit vector in place of fixed vector $\underline{a}$, i.e., $\underline{a} = \underline{r} = r\,\underline{i}$, where $\underline{i}$ is the unit vecor specifying the direction of increasing r. Thus

$$\underline{T}_n^m = \underline{L}_n^m \times \underline{r} = \text{curl}(r\psi_n^m) = \text{curl}(\underline{i}\, r \psi_n^m)$$

$$= \left(0, \frac{1}{\sin\theta} \frac{\partial \psi_n^m}{\partial \phi}, -\frac{\partial \psi_n^m}{\partial \theta}\right), \qquad (37)$$

and

$$\underline{S}_n^m = \frac{1}{\alpha} \text{curl}(\underline{T}_n^m) = \frac{1}{\alpha} \left\{ \frac{n(n+1)\psi_n^m}{r}, \right.$$

$$\left. \frac{1}{r} \frac{\partial^2 (r\psi_n^m)}{\partial r \partial \theta}, \frac{1}{r \sin\theta} \frac{\partial^2 (r\psi_n^m)}{\partial r \partial \phi} \right\}. \qquad (38)$$

The vectors $\underline{T}_n^m$ and $\underline{S}_n^m$ satisfy the orthogonal relations:

$$\int_0^{2\pi} \int_0^{2\pi} \underline{T}_n^m \underline{T}_{n'}^{m'} \sin\theta \, d\theta \, d\phi = 0, \text{ if } m = m', \; n = n', \qquad (39)$$

$$\int_0^{2\pi} \int_0^{2\pi} \underline{T}_n^m \underline{S}_{n'}^{m'} \sin\theta \, d\theta \, d\phi = 0 \text{ always}, \qquad (40)$$

$$\int_0^{\pi} \int_0^{2\pi} (T_n^m)^2 \sin\theta \, d\theta \, d\phi = \frac{2\pi}{(2n+1)} \frac{(n+m)!}{(n-m)!} n(n+1) Z_n^2(\alpha r), \tag{41}$$

and similar relations for $\underline{S}_n^m$

To analyze the field completely let us study the boundary conditions. At an interface where α is discontinuous the boundary conditions to be satisfied are the continuity of the normal resolutes of the electric current and the magnetic field. Using the notation of Section 4, Chapter 1, we have

$$\underline{J}_{1n} = J_{2n}, \; H_{1n} = H_{2n}. \tag{42}$$

Since $\underline{J} = \text{curl}\, \underline{H} = \alpha \underline{H}$, the relations (42) reduce to

$$(\alpha_1 - \alpha_2) H_n = 0. \tag{43}$$

Since $\alpha_1 = \alpha_2$, the boundary conditions for a force-free field reduce to

$$H_n = 0. \tag{44}$$

Thus, the normal component of magnetic field vanishes at any bounding surface.

If the force-free field is confined to the sphere r = a, substitution of the boundary condition (44) in the above-mentioned solution requires that $Z_n(a\alpha) = 0$. Therefore T_n^m is zero on the boundary.

We are now in a position to prove the equipartition of energy. Since $\underline{T}_n^m$ is orthogonal to $\underline{S}_n^m$, the total energy of the magnetic field within the sphere is the sum of the energies of the toroidal and poloidal fields. The energy of the poloidal field is

$$W_S = \frac{1}{2} \mu_e \int (S_n^m)^2 \, dV', \tag{45}$$

integrated through the volume of the sphere. But

$$\underline{S}_n^m = \frac{1}{\alpha} \text{ curl } (\underline{T}_n^m), \text{ div } \underline{T}_n^m = 0. \tag{46}$$

Therefore (45) becomes

$$W_S = \frac{1}{2} \mu_e \alpha^{-2} \int \left\{ \text{curl } \underline{T}_n^m \right\}^2 dV \tag{47}$$

The identity:

$$e^{ijk} T_{k,j} e^{imn} T_{n,m} = e^{ijk} (T_k e^{imn} T_{n,m}) - T_k e^{ijk} e^{imn} T_{n,mj}$$

gives the vector relation

$$(\text{curl } \underline{T}_n^m)^2 = \text{div} \left\{ \underline{T}_n^m \times \text{curl } \underline{T}_n^m \right\} + (\underline{T}_n^m) \cdot \text{curl curl } \underline{T}_n^m. \tag{48}$$

From (47), (48), the divergence theorem, and the condition that $T_n^m = 0$, on the boundary, we obtain

$$W_S = \frac{1}{2} \mu_e \alpha^{-2} \int \left\{ (T_n^m) \cdot \text{curl curl } (T_n^m) \right\} dV$$

$$= \frac{1}{2} \mu_e \int (T_n^m)^2 \, dV, \tag{49}$$

where we have used the relation curl curl $T_n^m = -\nabla^2 T_n^m = \alpha^2 T_n^m$. Thus

$$W_S = W_T. \tag{50}$$

8. Intrinsic relations for inviscid fluids

We have already discussed the contribution of the Lorentz force along the principal triad of the lines of force. In this section we deal with the behaviour of the flow and field quantities along the principal

triads of both streamlines as well as field lines. This entails a slight change of notation. In fact let the streamlines and the lines of force have the principal triads $\underline{t}_1$, $\underline{n}_1$, $\underline{b}_1$, and $\underline{t}_2$, $\underline{n}_2$, $\underline{b}_2$, respectively. Furthermore let κ_1 and σ_1 be, respectively, the curvature and torsion of the streamlines with κ_2 and σ_2 the corresponding quantities for the lines of force.

The momentum equation for steady, non-viscous and compressible flow is

$$\rho\, \underline{v} \cdot \operatorname{grad} \underline{v} = -\operatorname{grad} p + \mu_e\, \underline{J} \times \underline{H}. \tag{51}$$

If we take the scalar product of both sides of the equation (51) with $\underline{t}_1$, $\underline{n}_1$, and $\underline{b}_1$, respectively, and use Serret-Frenet formulae, we get

$$\rho\, v \cdot \frac{d\underline{v}}{ds_1} = -\frac{dp}{ds_1} + \mu_e\, (\underline{J} \times \underline{H})_{t_1}, \tag{52}$$

$$p\kappa_1\, v^2 = -\frac{dp}{dn_1} + \mu_e\, (\underline{J} \times \underline{H})_{n_1}, \tag{53}$$

$$0 = -\frac{dp}{db_1} + \mu_e\, (\underline{J} \times \underline{H})_{b_1}, \tag{54}$$

where $v^2 = \underline{v} \cdot \underline{v}$; and $\frac{d}{ds_1}$, $\frac{d}{dn_1}$ and $\frac{d}{db_1}$ denote the directional derivatives along $\underline{t}_1$, $\underline{n}_1$, and $\underline{b}_1$, respectively. Moreover, the quantity $(\underline{J} \times \underline{H})_{t_1}$ means the component of Lorentz force along the tangent to the streamlines and the same holds for the two other expressions of the same kind From (54) we observe that the rate of change of p along the binomials to the streamlines is equal to the corresponding component of Lorentz force.

To analyze the variation of flow and field quantities along the lines of force we write the momentum equation in the form

$$\mu_e \underline{H} \cdot \text{grad}\, \underline{H} = \text{grad}\, p^* + \rho\, \text{grad}\, (\tfrac{1}{2} v^2) + \underline{w} \times \underline{v}, \qquad (55)$$

where $\underline{w}$ is the vorticity vector, $p^* = p + \frac{\mu_e H^2}{2}$ is the total pressure and $H^2 = \underline{H} \cdot \underline{H}$.

If we take the scalar product of this equation with $\underline{t}_2$, $\underline{n}_2$, and $\underline{b}_2$, we have

$$\mu_e \underline{H} \cdot \frac{dH}{ds_2} \quad \frac{dp^*}{ds_2} + \rho \frac{d}{ds_2} (\tfrac{1}{2} v^2) + \rho(\underline{w} \times \underline{v})_{t_2} \qquad (56)$$

$$\mu_e \kappa_2 H^2 = \frac{dp^*}{dn_2} + \rho \frac{d}{dn_2} (\tfrac{1}{2} v^2) + \rho(\underline{w} \times \underline{v})_{n_2} \qquad (57)$$

$$O = \frac{dp^*}{db_2} + \rho \frac{d}{db_2} (\tfrac{1}{2} v^2) + \rho(\underline{w} \times \underline{v})_{b_2} . \qquad (58)$$

If the streamlines and the lines of force coincide, the relations (56-58) take a very simple form.

Now the equation of state is

$$d h = T\, dS + \frac{1}{\rho}\, d p, \qquad (59)$$

where h, S and T are the specific enthalpy, specific entropy and temperature, respectively.

Combining (51) and (59) we obtain

$$\rho\, (\underline{w} \times \underline{v}) = \rho\, T\, \text{grad}\, S \cdot \rho\, \text{grad}\, h_o + \mu_e\, (\underline{J} \times \underline{H}). \qquad (60)$$

where

$$h_o = h + \tfrac{1}{2} v^2 .$$

From equation (60) we can recover w_{n_1}, and w_{b_1}, the components of the vorticity normal to the streamlines. In fact if we set

$$\underline{w} = w_{t_1}\,\underline{t}_1 + w_{n_1}\underline{n}_1 + w_{b_1}\,\underline{b}_1,$$

the equation (60) gives

$$\rho\,\underline{v}\,(w_{n_1}\,\underline{b}_1 - w_{b_1}\,\underline{n}_1) = \rho\,T\,\text{grad}\,S - \rho\,\text{grad}\,h_o + \mu_e\,(\underline{J}\times\underline{H}). \tag{61}$$

Taking the scalar product of (61) with n_1 and b_1, we get w_{n_1} and w_{b_1}. The complete evaluation of $\underline{w}$ may now be effected as in the non-conducting gases.

Analogously, we have

$$(J_{n_2}\,\underline{b}_2 - J_{b_1}\,\underline{n}_2) = -\rho\,T\,\text{grad}\,S + \rho\,\text{grad}\,h_o + \rho(\underline{w}\times\underline{v}). \tag{62}$$

Let $\underline{N}$ be the common unit normal, at a point, to both the streamlines and the line of force passing through that point. Taking the scalar product of (60) with $\underline{N}$, yields

$$\rho\;\underline{w}\cdot(\underline{v}\times\underline{N})\;\rho\,T\,\frac{dS}{dn} - \rho\,\frac{dh_o}{dN} + \mu_e\,\underline{J}\cdot(\underline{H}\times\underline{N}). \tag{63}$$

If the streamlines and the lines of force are normal trajectories then $\underline{v}\times\underline{N} = \underline{vt}_2$ and $\underline{H}\times\underline{N} = H\underline{t}_1$ and the relation (63) becomes

$$\rho\,v\,w_{t_2} = \rho\,T\,\frac{dS}{dN} - \rho\,\frac{dh_o}{dN} - \mu_e\,H\,J_{t_1}. \tag{64}$$

Finally we attend to the field equation

$$\underline{v}\cdot\text{grad}\,\underline{H} - \underline{H}\cdot\text{grad}\,\underline{v} + \underline{H}\,\text{div}\,\underline{v} = 0. \tag{65}$$

We shall take the scalar product of (65) with $\underline{n}_1$ and $\underline{n}_2$ only. The scalar product with the other principal directions can be disposed of similarly.

Now

$$\underline{v} \cdot \operatorname{grad} \underline{H} \cdot \underline{n}_1 = v \frac{dH_{n_1}}{ds_1} + v H_{t_1} \kappa_1 - v H_{b_1} \sigma_1 . \tag{66}$$

while

$$\underline{H} \cdot \operatorname{grad} \underline{v} \cdot \underline{n}_1 = v H_{t_1} \kappa_1 + v H_{n_1} \frac{d\underline{t}_1}{dn_1} \cdot \underline{n}_1 - v H_{b_1} \frac{d\underline{t}_1}{d\underline{b}_1} \cdot \underline{n}_1 . \tag{67}$$

By taking the scalar product of (65) with $\underline{n}_1$, and using (66) and (67) we readily obtain

$$v H_{b_1} \sigma_1 = v \frac{dH_{n_1}}{ds_1} - v H_{n_1} \frac{dt_1}{dn_1} \cdot \underline{n}_1 - v H_{b_1} \frac{dt_1}{db_1} \cdot \underline{n}_1 + H_{n_1} \operatorname{div} \underline{v} . \tag{68}$$

Similarly, the scalar product of (65) with n_2 yields

$$v_{b_2} \sigma_2 = \frac{dH_{n_2}}{ds_2} - v_{n_2} \frac{d\underline{t}_2}{dn_2} \cdot \underline{n}_2 - v_{b_2} \frac{d\underline{t}_2}{db_2} \cdot n_2 . \tag{69}$$

Chapter III

Boundary Value Problems

In this chapter we shall discuss the effect of magnetic field on the classical flow problems such as Poiseulle, Couette, Stokes and Oseen's flows. Our main interest is the velocity and magnetic field distributions and their interactions.

1. Plane Poiseulle flow - Hartman solution

We consider the two-dimensional steady laminar flow of an incompressible, magnetized, viscous fluid between two parallel, straight, insulated walls at rest. The velocity of the flow is parallel to the walls which are in the direction of the x-axis. There is an external magnetic field of a uniform strength H_o in the z-direction. It is natural to assume that the fluid velocity is everywhere in the direction of the x-axis, and depends only on the z-coordinate and the time t. The pressure p, however, depends on x also because there must be a constant pressure gradient in the direction of motion in order to maintain a steady flow.

Regarding the magnetic field we observe that we should have $H_1 = H_x(z,t)$, $H_2 = 0$, and $H_3 = H_o + h(z,t)$. From the condition $\operatorname{div} \underline{H} = 0$, it follows that $\frac{dh}{dz} = 0$, with $h = 0$, at the boundarys. Therefore $h = 0$. To sum up we have

$$
\begin{aligned}
&v_1 = v_x(z,t), && v_2 = 0 = v_3, \\
&H_1 = H_x(z,t), && H_2 = 0, \ H_3 = H_o, \qquad (1) \\
&p = p(x,z,t).
\end{aligned}
$$

The flow and field equations are

$$\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \operatorname{grad} \underline{v} = -\operatorname{grad}\left(p + \mu_e \frac{H^2}{2}\right) \mu_e \nabla^2 \underline{v} + \mu_e \underline{H} \cdot \operatorname{grad} \underline{H}, \tag{2}$$

$$\frac{\partial \underline{H}}{\partial t} + \underline{v} \cdot \operatorname{grad} \underline{H} - \underline{H} \cdot \operatorname{grad} \underline{v} = \eta \nabla^2 \underline{H}, \tag{3}$$

when simplified by the relations (1), they yield

$$\mu_e H_o \frac{dH_x}{dz} + \mu \frac{d^2 v_x}{dz^2} = \frac{dP}{dx}, \tag{4}$$

$$H_o \frac{dv_x}{dz} + \eta \frac{d^2 H_x}{dz^2} = 0, \tag{5}$$

and the relation $p + \frac{H_x^2}{2} = P(x)$, where $P(x)$ is a function of x only. The pressure gradient $-\partial P/\partial x = -\frac{dP}{dx}$, is constant in the x-direction, as pointed out before, for maintaining the steady flow.

Boundary conditions. We consider the two plates situated at $z = \pm a$; the plane $z = 0$, lies half way between them. No-slip condition for viscous flow gives $v_x = 0$, for $z = \pm a$. The magnetic field must satisfy the conditions $H_x = 0$, for $z = \pm a$ since the magnetic field outside the fluid is just H_o, and the tangential component of the magnetic field is continuous at the boundary. To sum up we have

$$v_x = H_x = 0, \quad \text{at } z = \pm a \tag{6}$$

To solve the equations (4) and (5) subject to the boundary conditions (6), we put them in non-dimensional form (see Section 5, Chapter 1). Put

$$v_x = U v'_x, \quad H_x = H_o H'_x,$$

$$p = \rho U^2 p'$$

The system (4) and (5) becomes, after dropping the primes,

$$\frac{M^2}{R_m} \frac{dh_x}{dz} + \frac{d^2 v_x}{dz^2} = \frac{dP}{dx} \quad (= \text{a constant},) \tag{7}$$

and

$$\frac{dv_x}{dz} + R_m \frac{d^2 H_x}{dz^2} = 0, \tag{8}$$

Where M and R_m are the Hartman and magnetic Reynolds numbers as defined in Section 5 of Chapter 1, i.e., $M^2 = \mu_e H_o^2 a^2(\sigma/\mu)$, $R_m = UL\, \mu_e \sigma$.

By differentiation, we get from (1) and (8):

$$\frac{d^3 v_x}{dz^3} - M^2 \frac{dv_x}{dz} = 0. \tag{9}$$

A solution of equation (9) is

$$v_x = \frac{A}{M} \cosh Mz + \frac{B}{M} \sinh Mz + C. \tag{10}$$

With the help of boundary conditions (6), an appropriate solution for v_x can be readily found and then by going back to (8), the value of H_x can be calculated. The results for v_x and H_x, in physical units, are

$$v_x = v_o \frac{\cosh M - \cosh (Mz/a)}{\cosh M - 1} \tag{11}$$

$$H_x = (\mu/\eta)^{\frac{1}{2}} v_o \frac{\frac{z}{a} \sinh M - \sinh (M z/a)}{\cosh M - 1}, \tag{12}$$

where $v_o = - \frac{dP}{dx} \frac{a^2}{\mu} \frac{\cosh M - 1}{M \sinh M}$, is the velocity along $z = 0$.

The fluid velocity averaged over the cross-section is

$$\bar{v} = \frac{1}{2a} \int_{-a}^{a} v_x \, dz = - \frac{dP}{dx} \frac{a^2}{\mu M} \left(\coth M - \frac{1}{M} \right), \quad (13)$$

For $M \ll 1$, we have

$$v_x = \frac{v_o}{U} \left(1 - \frac{z^2}{a^2} \right), \quad \bar{v} = - \frac{dP}{dx} \frac{a^2}{3\mu} , \quad (14)$$

which agree with the results of ordinary fluid dynamics.

For $M > > 1$,

$$v_x = v_o \left(1 - e^{(1 - |z| a)M} \right) ,$$

$$\bar{v} = - \frac{dP}{dx} \frac{a}{H_o} \sqrt{\frac{\eta}{\mu}} . \quad (15)$$

When the magnetic field increases, the velocity profile is flattened over the major part of the cross section, and the mean velocity is reduced (for a given pressure gradient). The flow pattern is illustrated in Figure 1.

2. Plane Couette flow

This is a slight variation of the problem discussed in article 1. The two plates are still situated at $z = \pm a$. However, there is a relative velocity 2U between them and there is no pressure gradient in the flow field. Thus the boundary conditions in dimensionless units are

$$z = + 1, \; v_x = 1; \; z = -1, \; v_x = - 1, \; z = 0, \; v_x = 0. \quad (16)$$

The solution of the differential equation (9) appropriate for the present boundary conditions is

$$v_x = \frac{\sinh M z}{\sinh M} \tag{17}$$

As $M \to 0$, we have the velocity distribution of Couette flow in ordinary hydrodynamics, viz; $v_x = z$. When $M \to \infty$, the relation (17) gives $v_x \to 0$, except at $z = \pm 1$, where, of course, $v_x = \pm 1$, respectively. Here again (see Figure 2) we have a uniform flow between two plates and a very large velocity gradient near the walls.

The values of H_x and p can be easily obtained by the procedure outlined in article 1. Since the plates are insulated and originally $H_x = 0$, on the plates $z = \pm 1$, H_x should remain zero always on the plates.

Let us observe in passing that if the external magnetic field is in the direction of the plates instead of perpendicular to the plates, the velocity distributions of the plane Poiseulle flow and plane Couette flow will not be affected by the magnetic field.

3. Rayleigh's problem: - unsteady motion of a plate

Let an infinite flat insulated plate be set impulsively into uniform motion with velocity U in its own plane in the presence of a transverse uniform magnetic field of strength H_o as shown in figure 3. The geometry of the flow requires that (c.f. the analysis of article 1)

$$\begin{aligned} &v_x = v_x(z,t),\ v_y = v_z = 0,\ p = \text{a constant}, \\ &H_x = H_x(z,t),\ H_y = 0,\ H_z = 0. \end{aligned} \tag{18}$$

The flow and field equations (2) and (3) become

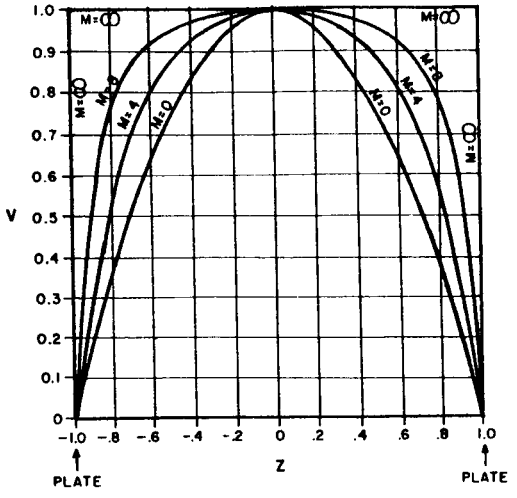


FIGURE I. VELOCITY DISTRIBUTIONS OF PLANE POISEUILLE FLOW IN MAGNETOHYDRODYNAMICS

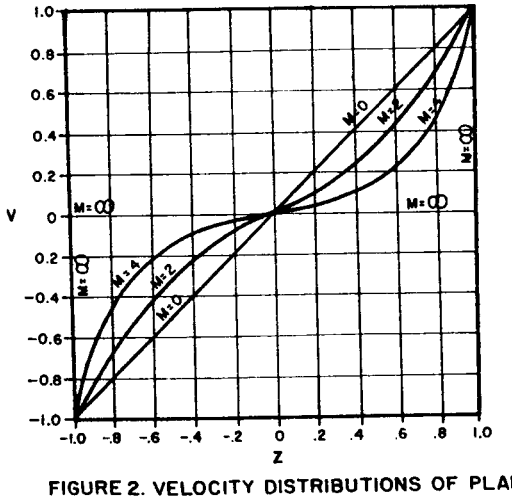


FIGURE 2. VELOCITY DISTRIBUTIONS OF PLANE COUETTE FLOW IN MAGNETOHYDRODYNAMICS

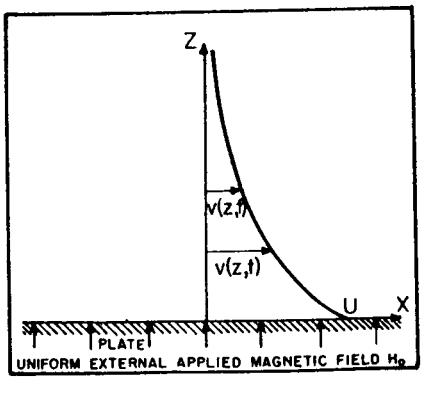


FIGURE 3. RAYLEIGH PROBLEM IN MAGNETOHYDRODYNAMICS

$$\rho \frac{\partial v_x}{\partial t} - \mu_e H_o \frac{\partial H_x}{\partial z} - \mu \frac{\partial^2 v_x}{\partial z^2} = 0, \tag{19}$$

$$\frac{\partial H_x}{\partial t} - H_o \frac{\partial v_x}{\partial z} - \eta \frac{\partial^2 H_x}{\partial z^2} = 0. \tag{20}$$

The initial conditions are

$$t = 0, \; v_x = H_x = 0, \tag{21}$$

and the boundary conditions are

$$\begin{aligned} z &= 0, \; v_x = U, \; H_x = 0, \\ z &= \infty, \; v_x = 0, \; H_x = 0. \end{aligned} \tag{22}$$

By the method of Laplace transform, the required solution to this problem can be obtained in principle. The solution thus obtained is very complicated since its exact inversion would be extremely difficult. However, an approximate solution for very small electrical conductivity can be given. In fact let us assume that

$$H_o >> 1, \quad \eta = (\mu e \sigma)^{-1} >> 1,$$

so that the field equation (20) is replaced by

$$H_o \frac{\partial v_x}{\partial z} - \frac{\partial^2 H_x}{\partial z^2} = 0, \tag{23}$$

which, by quadrature and because of the boundary conditions, gives

$$\frac{\partial H_x}{\partial z} = - \frac{1}{\eta} H_o v_x . \tag{24}$$

The equation (19), when simplified with the help of (24), becomes

$$\frac{\partial v_x}{\partial t} + \frac{\mu_e^2 \sigma H^2}{\rho} v_x = \nu \frac{\partial^2 v_x}{\partial z^2} , \tag{25}$$

while the boundary conditions (22) remain unchanged. If we set

$$u = \int_0^\infty e^{-st} \, v_x \, dt,$$

the Laplace Transform of (25) gives

$$\nu \frac{d^2 u}{dz^2} = u \, (s + m),$$

where

$$m = (\mu_e^2 \, \sigma \, H_o^2)/\rho. \tag{26}$$

The required solution of (26), with the given boundary and initial conditions, is

$$u = \frac{U}{s} \, e^{-\left\{ (s + m)^{\frac{1}{2}} \frac{z}{\sqrt{\nu}} \right\}} \tag{27}$$

The inverse of (27) gives, finally, the velocity distribution

$$u(z,t) = \frac{U}{2} \left\{ e^{-z\sqrt{\frac{m}{\nu}}} \, \operatorname{erfc} \left(\frac{z}{2\sqrt{\nu t}} - m t \right) + e^{z\sqrt{\left(\frac{m}{\nu} \right)}} \right.$$

$$\left. \operatorname{erfc} \left(\frac{z}{2\sqrt{\nu t}} + mt \right) \right\}, \tag{28}$$

where the function erfc denotes the complimentary error function defined as 1 - erf. When $m \to 0$, the equation (28) reduces to the classical solution of hydrodynamics.

4. Stokes flow

(1) Translational motion of an axially-symmetric body

The streaming motion of an incompressible, viscous, electrically non-conducting fluid past an axially-symmetric body for small Reynolds

numbers is well known. In this section we discuss the case when the fluid is magnetized. The flow is assumed to be steady and parallel to the negative x-axis at infinity. The aligned magnetic field H_o at infinity, is uniform and is in the direction of the flow. The field and flow equations in physical units are

$$\begin{aligned} \operatorname{div} \underline{H} &= 0, \\ \operatorname{curl} (\underline{v} \times \underline{H}) &= \eta \operatorname{curl} \operatorname{curl} \underline{H}; \\ \operatorname{div} \underline{v} &= 0, \\ \rho (\underline{v} \cdot \operatorname{grad}) \underline{v}' &= - \operatorname{grad} p + \mu \nabla^2 v + \mu_e \underline{J} \times \underline{H}. \end{aligned} \tag{29}$$

Let U be the speed of the uniform stream of infinity and a be a characteristic length of body. The space coordinates may then be made dimensionless with the factor a^{-1} and the pressure, velocity and magnetic field as

$$p' = \frac{a}{\mu U} p, \quad \underline{v}' = \frac{\underline{v}}{U}, \quad \underline{H}' = \frac{\underline{H}}{H_o}, \tag{30}$$

The boundary conditions are that $\underline{v}' = \underline{i}$ (unit vector along the x-axis) at inifinity and $\underline{v}' = 0$ at the body. The field equation (29) becomes, after dropping the primes,

$$R_m \operatorname{curl} (\underline{v} \times \underline{H}) = \operatorname{curl} \operatorname{curl} \underline{H}, \tag{31}$$

where R_m is the magnetic Reynolds number. If we assume that R_m is negligible, this equation implies that the magnetic field is independent of the fluid velocity. We may, therefore, take the magnetic field to be uniform and parallel to the x-axis at all points. This makes $\underline{v} \times \underline{H} = 0$ both at the sphere and at infinity. The field equations curl $\underline{E} = 0$, and $\underline{J} = 0$

($\underline{E} + \mu_e \ \underline{v} \times \underline{H}$), as well as the boundary conditions on $\underline{E}$ and $\underline{J}$ are satisfied by setting

$$\underline{E} = 0, \quad \underline{J} = U \ \mu_e \ \sigma \ \underline{v} \times \underline{H}. \tag{32}$$

We are thus left with the flow equations which, in view of (30) and (32), become (primes omitted)

$$\operatorname{div} \underline{v} = 0,$$

$$R \ (\underline{v} \cdot \operatorname{grad}) \ \underline{v} = \nabla p + \nabla^2 \underline{v} + M^2 \ (\underline{v} \cdot \underline{i} - \underline{v}), \tag{33}$$

where R and M are, respectively, the Reynolds and Hartman numbers. If we also assume R to be small the system (33) reduces to

$$\operatorname{div} \underline{v} = 0,$$

$$-\nabla p + \nabla^2 \underline{v} - M^2 \ \underline{v} - [\ (\underline{v} \cdot \underline{i})\ ] = 0. \tag{34}$$

We consider the following expansions of the exact solutions of (34):

$$\underline{v} = h^o \ (x, y, z) + M \ \underline{h}^{(1)} \ (x, y, z) + M^2 \ \underline{h}^{(2)} \ (x, y, z) + \ldots \ldots,$$

$$p = p^o \ (x, y, z) + M \ p^{(1)} \ (x, y, z) + M^2 \ p^{(2)} \ (x, y, z) + \ldots \ldots \tag{35}$$

By insertion of the above expansions into equation (34), the following equations result:

$O(1)$;

$$\operatorname{div} \underline{h}^o = 0,$$

$$-\nabla p^{(o)} + \nabla^2 \underline{h}^{(o)} = 0; \tag{36}$$

$O(M)$;

$$\operatorname{div} \underline{h}^{(1)} = 0,$$

$$\nabla p^{(1)} + \nabla^2 \underline{h}^{(1)} = 0; \tag{37}$$

$O(M^2)$;

$$\operatorname{div} \underline{h}^{(2)} = o,$$

$$-\nabla p^{(2)} + \nabla^2 \underline{h}^{(2)} = \underline{h}^{(o)} - (\underline{h}^{(o)} \cdot \underline{i}) \ \underline{i}; \tag{38}$$

etc. Equations (36) and (37) are identical to the classical Stokes equations. This indicates that at finite distance from the body, i.e., $r(= \sqrt{x^2 + y^2 + z^2}) = 0(1)$, in the limit $M \to 0$, the flow field is essentially governed by Stokes' equations. However, it can be shown that at great distances from the body (i.e., $r = 0(M^{-\alpha})$, $0 < \alpha < 1$) the above expansions are not uniform approximations of the exact solutions. By not being uniform approximations we mean that the difference between the expansions and the exact solutions for a given value of M becomes arbitrarily large as $r \to \infty$. A uniformly valid approximation of the exact solution in the neighborhood of $r \to \infty$ is given by a second type of expansions using different variables, the so-called outer-expansions. The expansions defined by (35) are called inner expansions. The outer expansions are defined as

$$\underline{v} = \underline{i} + M\,\underline{g}^{(1)}(\tilde{x},\tilde{y},\tilde{z}) + M^2\,g^{(2)}(\tilde{x},\tilde{y},\tilde{z}) + \ldots .$$

$$p = M^2\tilde{p}^{(1)}(\tilde{x},\tilde{y},\tilde{z}) + M^3\,p^{(2)}(\tilde{x},\tilde{y},\tilde{z}) + \ldots . \qquad (39)$$

and the independent variables are now (outer variables)

$$\tilde{x} = M\,x,\ \tilde{y} = My,\ \tilde{z} = mz. \qquad (40)$$

The above expansions may be formally obtained from the exact solutions by the limit process $M \to 0$ with fixed x, y, z and are expected to be uniformly valid for all $r = O(M^{-1})$.

Inserting (39) into (34) yields

$$-\nabla p^{(j)} + \nabla^2 \underline{g}^{(j)} - \left\{ \underline{g}^{(j)} - \underline{g}^{(j)} \cdot \underline{i} = 0 \right\} \quad (j = 1, 2, \ldots) \qquad (41)$$

These equations are identical to the exact equations; however, as we shall soon see, the boundary conditions for these equations are simpler than those for the equations (34).

Since the inner expansions are not valid at infinity, boundary condition:

$$\underline{v} \to \underline{i}, \text{ as } r \to \infty;$$

does not, in general, hold for the inner expansions. For a similar reason, boundary condition: $\underline{v} = 0$, at the boundary, does not, in general, hold for the outer expansions. These boundary conditions are replaced by the matching conditions, which have the requirements that the inner and the outer expansions should agree, term for term, for some intermediate orders of r, namely $r = O(M^{-\alpha})$, $0 < \alpha < 1$, for which the inner and outer expansions are both valid. The use of the matching conditions will become clear from the way in which the solutions are determined.

The first term in the outer expansion $(39)_1$ is the free-stream velocity $\underline{v} = \underline{i}$. This term may be understood by the following reasoning. Assume the solid is a sphere of radius a. Then the boundary of the solid is given in outer variables by $r = M$ and in the limit $M \to 0$, the body shrinks to zero. A point cannot cause a finite disturbance in the fluid, hence the value of $\underline{v}$ will tend to the free-stream velocity $\underline{i}$.

For the inner solutions, the no-slip conditions are valid and therefore $\underline{h}^{(i)} = 0$, at the body for all i, in particular for $i = 0$. On the other hand by matching conditions, $\underline{h}^{(0)}$ must agree for large values of r with the leading term of the outer expansion and hence

$$\underline{h}^{(0)} \to \underline{i}, \text{ as } r \to \infty \ . \tag{42}$$

From equations (36) and these boundary conditions for $\underline{h}^{(0)}$, we find that the solution of $\underline{h}^{(0)}$ and $\underline{p}^{(0)}$ are simply the solutions of the non-magnetic Stokes problems. For a sphere such solutions are

$$h^{(0)} = i - \frac{3}{2}\left(\frac{\underline{i}}{r} - \nabla\frac{x}{2r}\right) + \frac{1}{4}\left(\nabla \frac{\partial}{\partial x}\frac{1}{r}\right),$$

$$p^{(0)} = -\frac{3x}{2r^3} \ . \tag{43}$$

The drag on the sphere in physical units is $6\pi\mu U a$.

For large values of r the asymptotic expansions of the Stokes solutions for axially symmetric bodies of any shape are given by

$$\underline{h}^{(0)} = \underline{i} - \frac{D_o}{4\pi}\left(\frac{\underline{i}}{r} - \nabla\frac{x}{2r}\right) + O\left(\frac{1}{r^2}\right),$$

$$p^{(0)} = -\frac{D_o x}{4\pi r^3} + O\left(\frac{1}{r^3}\right), \tag{44}$$

where D_o is the non-magnetic Stokes drag for the corresponding body. If one defines $r_\alpha = M^2 r$, where $0 < \alpha < 1$, and rewrites (44) in this variable, one gets

$$\underline{h}^{(0)}(x_\alpha, r_\alpha; M) = \underline{i} - \frac{D_o}{4\pi}\left(\frac{\underline{i}}{r_\alpha} - \nabla_\alpha \frac{x}{2r_\alpha}\right) + O(M^{2\alpha}),$$

$$p^{(0)}(x_\alpha, r_\alpha; M) = -\frac{D_o}{4\pi}\frac{x_\alpha}{r_\alpha^3} M^{2\alpha} + O(M^{2\alpha}). \tag{45}$$

Equations (45) imply that, when r is of an intermediate order $O(M^{-\alpha})$,

$0 < \alpha < 1$, in comparison with that used in the inner ($\alpha = 0$) and outer ($\alpha = 1$) limits, the leading terms of the expansion of $\underset{\sim}{h}^{(0)}$ and p^{0} are the terms displayed in (44). The first-order solutions of the outer expansions, $\underset{\sim}{g}^{(1)}$ and $\tilde{p}^{(1)}$ must, therefore, agree with the inner solutions in these terms.

We now find the first-order outer solutions $g^{(1)}$ and $\tilde{p}^{(1)}$. They are determined from the equations (39) with the boundary conditions $g^{(1)} \to 0$ as $r \to \infty$ and the matching condition stated above. Such solutions are given by

$$\underset{\sim}{g}^{(1)} = D_o \underset{\sim}{G}, \tilde{p}^{(1)} = D_o P, \tag{46}$$

where

$$G = -\frac{1}{4\pi}\left[\frac{i}{\tilde{r}} e^{-\frac{1}{2}\tilde{r}} \cosh\frac{\tilde{x}}{2} \cdot \tilde{\nabla}\frac{1}{r} e^{-\frac{1}{2}\tilde{r}} \sinh\frac{\tilde{x}}{2}\right],$$

$$P = \frac{1}{4\pi}\left[\frac{\partial}{\partial \tilde{x}} \frac{1}{\tilde{r}} e^{-\frac{1}{2}\tilde{r}} \cosh\frac{\tilde{x}}{2} - \frac{1}{r} e^{-\frac{1}{2}\tilde{r}} \sinh\frac{\tilde{x}}{2}\right]. \tag{47}$$

When (47) are rewritten in the variable r_α and expanded in powers of M, there are obtained the results

$$M\underset{\sim}{G} = -\frac{1}{4\pi}\left(\frac{i}{r_\alpha} - \nabla_\alpha \frac{x_\alpha}{2r_\alpha}\right) M^\alpha + \frac{i}{16\pi} M + O(M^{2-\alpha}),$$

$$M^2 P = -\frac{x_\alpha}{4\pi r_\alpha^3} M^{2\alpha} + O(M^2), \tag{48}$$

By comparing these results with (45), we see that the inner and outer expansions are matched to the order $O(M^\alpha)$. The leading term in the outer expansion of $\underset{\sim}{v}$ which is not matched by the inner expansion of $\underset{\sim}{v}$ is of order of O(M). From (48), it follows that the second-order solution

$\underline{h}^{(1)}$ which satisfied the Stokes equations (37) should satisfy the boundary condition $\underline{h}^{(1)} = 0$, at the body and the condition $\underline{h}^{(1)} = \frac{D_o \underline{i}}{16\pi}$ at $r \to \infty$. Such a solution is easily seen to be

$$\underline{h}^{(1)} = \frac{D_o}{16\pi} \underline{h}^{(0)} \tag{49}$$

The corresponding pressure is

$$p^{(1)} = \frac{D_o}{16\pi} p^{(0)} . \tag{50}$$

To the order O(M), the inner solution of the velocity field is then given as

$$\underline{v} = (1 + \frac{D_o}{16\pi} M) \underline{h}^{(0)} + O(M^2), \tag{51}$$

which shows that the effect of the magnetic field on the inner Stokes flow is an apparent increase of the free-stream velocity; in the ratio $1 : (1 + \frac{D_o}{16\pi} M)$. There is a corresponding amount of increase in the drag on the body, as can be easily verified by a simple argument. The drag on the body is, therefore, given by the formula (reverting to physical units)

$$D = D_o (1 + \frac{D_o}{16\pi\mu a U} M) + O(M^2). \tag{52}$$

The value of D can now be calculated for all the bodies for which the Stokes drag D_o is known. For example, for a sphere and disk (both of radius a) the values of D_o are $6\pi\mu aU$ and $16\mu a U$. The corresponding values of D are, respectively, then $6\pi \mu aU (1 + \frac{3}{8} M)$, and $16 \mu a U (1 + \frac{1}{\pi} M)$. For a sphere the terms of higher order then M have also been found, namely

$$D = D_o \left[1 + \frac{3}{8} M + \frac{7}{960} M^2 - \frac{43}{7680} M^3 + O(M^4)\right] \qquad (53)$$

(2) Rotary motion of an axially symmetric body:

We now study the effect of a uniform magnetic field on Stokes flow when an axially symmetric body is uniformly rotating about its axis of asymmetry.

It is clear that for small magnetic Reynolds numbers, we shall get the same equations of motion as in the previous section. However, now the characteristic velocity may be taken as the uniform angular velocity of the rotating body multiplied by the characteristic length of the body.

Let x be the axis of symmetry as before and y and z axes be mutually orthogonal axes in a plane perpendicular in the x-axis. Then the boundary conditions are

$$v_1 = 0, \quad v_2 = \omega_o z, \quad v_3 = -\omega_o y, \qquad (54)$$

where ω_o is the uniform angular velocity of the body.

Let ω be the angular velocity of the flow field and let the quantity Ω stand for $r^2\omega$, where r^2 is now equal to $y^2 + z^2$. Equations (34_1) and (54) are satisfied if we set

$$v_1 = 0, \quad v_2 = \frac{z}{r^2}\,\Omega, \quad v_3 = \frac{-y}{r^2}\,\Omega, \qquad (55)$$

p = constant,

while equation (34_2) becomes

$$(D^2 - M^2)\,\Omega = 0, \qquad (56)$$

$$D^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} \quad . \tag{57}$$

It is possible to get an exact solution of equation (56) for a sphere. In fact, in spherical polar coordinates the operator D^2 becomes

$$D^2 = \frac{\partial^2}{\partial R^2} + \frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial r}{\partial\theta}\right).$$

The appropriate solution of equation (56), which now takes the form:

$$\frac{\partial^2\Omega}{\partial R^2} + \frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Omega}{\partial\theta}\right) - M^2\Omega = 0, \tag{58}$$

$$\Omega = A\, e^{-MR}\left(1 + \frac{1}{MR}\right)\sin^2\theta \, . \tag{59}$$

Therefore

$$\omega = \frac{\Omega}{R^2\sin^2\theta} = A\,\frac{e^{-MR}}{R^2}\left(1 + \frac{1}{MR}\right). \tag{60}$$

Since $\omega = \omega_o$ at $R = a$ (the radius of the sphere), we obtain

$$\omega = \frac{M\omega_o a^3 e^{Ma}}{1 + Ma}\,\frac{e^{-MR}}{R^2}\left(1 + \frac{1}{MR}\right). \tag{61}$$

The frictional force T on the unit area of the sphere is

$$T = \mu\omega o\,\frac{(3 + 3\,Ma + M^2a^2)}{1 + Ma}\,\sin\theta. \tag{62}$$

The total couple on the sphere is

$$N = \int_o^{\pi} T\, a \sin\theta \cdot 2\pi a^2 \sin\theta d\theta = \frac{8}{3}\pi\mu a^3\omega_o\,\frac{3 + 3\,Ma + M^2a^2}{1 + Ma}\,, \tag{63}$$

When $M \to 0$, $N \to N_o = 8\pi\mu a^3\omega$, which is the known value of the couple for non-conducting fluids.

From (63) it follows that for small M, $N = N_o + O(M^2)$, i.e., the magnetic field has no first order effect on the couple. If we employ the expansion procedures of the last section, it would emerge, that for small M, the magnetic field has no first order effect on the couple experienced by an axially symmetric body in general.

5. Oseen flow

In the study of Stokes flow we neglected the distrubance of the magnetic field and the inertia of the fluid. This amounted to assuming that the Reynolds number, R, and magnetic Reynolds number R_m, are small compared to the Hartman number, M, though the latter is itself supposed to be small. In this section we take into account, to some extent, the effects previously neglected by using an Oseen-type approximation in which quadratic terms in the disturbance quantities are neglected in the equations of motion. Only the case of a sphere is discussed.

It emerges that the ratio $\mu_e H_o^2/\rho U^2 = M^2/RR_m$ of magnetic pressure to dynamic pressure is an important parameter in the discussion. For $M^2 < RR_m$, the two wakes which appear both lie downstream of the sphere, while, for $M^2 > RR_m$, one lies upstream; and the drag experienced by the sphere depends in a completely different way on M, R and R_m in the two cases.

As in the previous section we write the equations (29) in dimensionless form by referring $\underline{v}$, $\underline{x}$ and $\underline{H}$ to the velocity at infinity U, the radius of the sphere, a, and the external field intensity H_o, the equations

reduce to

$$\operatorname{div} \underline{H} = 0$$

$$\operatorname{curl} \operatorname{curl} \underline{H} = R_m \operatorname{curl} \underline{v} \times \underline{H},$$

$$\operatorname{div} \underline{v} = 0$$

$$R \underline{v} \cdot \operatorname{grad} \underline{v} = -\operatorname{grad} p + \nabla^2 \underline{v} + \frac{M^2}{R_m} \operatorname{curl} \underline{H} \times \underline{H}, \quad (64)$$

where we have used the relation $\underline{J} = \operatorname{curl} \underline{H}$. The electric field $\underline{E} = 0$, as in the previous section because of the geometry of the flow. In fact, for an axially symmetric motion in which the velocity $\underline{v}$ and the magnetic field $\underline{H}$ lie in the meridian plane and are independent of the azimuthal angle, the conduction equation $\operatorname{curl} \underline{H} = \sigma (\underline{E} + \mu_e \underline{v} \times \underline{H})$ shows that the electric field E is perpendicular to this plane and also independent of the azimuthal angle. Relation $\operatorname{curl} \underline{E} = 0$, then shows that $\underline{E} = 0$, if it is to be finite on the axis.

If the x-axis is taken along the axis of symmetry, then at infinity $\underline{v} = \underline{H} = \underline{i}$. Following Oseen's work for the purely viscous case, we linearize these equations by writing $\underline{v} + \underline{i}$ for $\underline{v}$ and $\underline{h} + \underline{i}$ for $\underline{H}$, and neglecting quadratic terms in $\underline{v}$ and $\underline{h}$. From $(64)_4$ we then find

$$R \frac{\partial \underline{v}}{\partial x} = -\operatorname{grad} p + \nabla^2 \underline{v} + \frac{M^2}{R_m} \left\{ (\operatorname{curl} h) \times \underline{i} \right\}. \quad (65)$$

But

$$(\operatorname{curl} \underline{h}) \times \underline{i} = (\underline{i} \cdot \operatorname{grad}) \underline{h} - \operatorname{grad} (\underline{h} \cdot \underline{i})$$

Therefore the equation (65) becomes

$$\nabla^2 \underline{v} - R \frac{\partial \underline{v}}{\partial x} + \frac{M^2}{R_m} \frac{\partial \underline{h}}{\partial x} = \text{grad}\left(p + \frac{M^2}{R_m} \underline{h} \cdot \underline{i}\right). \qquad (66)$$

Similarly the equation $(64)_2$ yields

$$\nabla^2 \underline{h} + R_m \left(\frac{\partial \underline{v}}{\partial x} - \frac{\partial \underline{h}}{\partial x} \right) = 0. \qquad (67)$$

From (66) we have

$$\nabla^2 \left(p + \frac{M^2}{R_m} \underline{h} \cdot \underline{i}\right) = 0;$$

and a particular integral of equalities (66) is, therefore, obtained if we write

$$R_m p + M^2 \underline{h} \cdot \underline{i} = (M^2 - RR_m) \frac{\partial \phi}{\partial x},$$

$$\underline{v} = \underline{h} = \text{grad}\, \phi,$$

where ϕ satisfies

$$\nabla^2 \phi = 0. \qquad (68)$$

This solution is completed if we set

$$\underline{v} = \text{grad}\, \varphi + \underline{v}', \qquad \underline{h} = \text{grad}\, \varphi + \underline{h}',$$

where $\underline{v}'$, $\underline{h}'$ are solutions of the corresponding homogeneous equations:

$$\nabla^2 \underline{v}' - R \frac{\partial \underline{v}}{\partial x} + \frac{M^2}{R_m} \frac{\partial \underline{h}'}{\partial x} = 0, \qquad (69)$$

$$\nabla^2 \underline{h}' + R_m \left(\frac{\partial \underline{v}'}{\partial x} - \frac{\partial \underline{h}'}{\partial x} \right) = 0, \qquad (70)$$

Multiply (70) by α (to be soon determined) and add to (69), getting

$$\nabla^2 (\underline{v}' + \alpha \underline{h}') + (-R + \alpha R_m) \frac{\partial}{\partial x} \left\{ \underline{v}' + \frac{M^2 - \alpha R_m^2}{(-R + \alpha R_m) R_m} \underline{h}' \right\} = 0. \qquad (71)$$

Let α be such that

$$\frac{M^2 - \alpha M_m^2}{(-R + \alpha R_m) R_m} = \alpha, \tag{72}$$

which is a quadratic algebraic equation in α giving two roots α_1, and α_2 as

$$\alpha_1, \alpha_2 = \frac{1}{2R_m} \left\{ R - R_m \pm \sqrt{(R-R_m)^2 + 4 M^2} \quad . \right\} \tag{73}$$

Equation (71) can then be written as

$$\begin{aligned} (\nabla^2 - 2 k_1 \frac{\partial}{\partial x}) (\underline{v'} + \alpha_1 \underline{h'}) &= 0, \\ (\nabla^2 - 2 k_2 \frac{\partial}{\partial x}) (\underline{v'} + \alpha_2 \underline{h'}) &= 0, \end{aligned} \tag{74}$$

where

$$k_1, k_2 = \frac{1}{4} \left\{ R + R_m \mp (R - R_m)^2 + 4 M^2 \right\} \quad .$$

In addition we clearly have

$$\text{div } (\underline{v'} + \alpha_1 \underline{h'}) = \text{div } (\underline{v'} + \alpha_2 \underline{h'}) = 0. \tag{75}$$

The equations (74) and (75) are now the same equations as are satisfied by the velocity vector in classical Oseen flow. Their solution is well known and for the present case leads to (Lamb's Hydrodynamics)

$$\begin{aligned} \underline{v'} + \alpha_1 \underline{h'} &= (\alpha_2 - \alpha_1) \left\{ \frac{1}{2k_1} \text{ grad } \chi_1 - \chi_1 \underline{i} \right\} \\ \underline{v'} + \alpha_2 \underline{h'} &= (\alpha_2 - \alpha_1) \left\{ \frac{-1}{2k_2} \text{ grad } \chi_2 - \chi_2 \underline{i} \right\}, \end{aligned} \tag{76}$$

where

$$(\nabla^2 - 2k_1 \frac{\partial}{\partial x}) \chi_1 = (\nabla^2 - 2k_2 \frac{\partial}{\partial x}) \chi_2 = 0. \qquad (77)$$

Therefore the complete solution is

$$\underline{v} = \text{grad}\,(\varphi + \frac{2}{2k_1} \chi_1 - \frac{\alpha_1}{2k_2} \chi_2) - (\alpha_2 \chi_1 - \alpha_1 \chi_2)\underline{i},$$

$$\underline{h} = \text{grad}\,(\varphi - \frac{1}{2k_1} \chi_1 + \frac{1}{2k_2} \chi_2) + (\chi_1 - \chi_2)\underline{i},$$

$$p = -R \frac{\partial \varphi}{\partial x} + \frac{M^2}{R_m} \left\{ \left(\frac{1}{2k_1} \frac{\partial \chi_1}{x} - \frac{1}{2k_2} \frac{\partial \chi_2}{x} \right) - (\chi_1 - \chi_2) \right\}. \qquad (78)$$

Inside the sphere $\text{curl}\,\underline{h} = \text{div}\,\underline{h} = 0$, and hence we may set

$$\underline{h} = \text{grad}\ \psi\ , \qquad (79)$$

where

$$\nabla^2 \psi = 0. \qquad (80)$$

The fundamental solutions of (77), from which the general solutions may be constructed, are $\exp\left\{ - \left| k_1 \right| r + k_1 x \right\}/r$ and $\exp\left\{ - \left| k_2 \right| r + k_2 x \right\}/r$, respectively. Now k_2 is essentially positive, so that there is always a downstream wake. On the other hand, we have $k_1 \gtrless 0$ according as $M^2 \gtrless RR_m$. Thus, when μH_o^2 is less than ρU^2, the second wake is also downstream, but when it is greater the wake is upstream.

While discussing the Stokes flow we had just the velocity field to calculate, as such only the boundary conditions related to $\underline{v}$ came into play. We now have to calculate the magnetic field as well. For simplicity we assume that the permeability of the sphere is the same as that of the fluid. Therefore, the magnetic field $\underline{h}$ must be continuous

at the surface of the sphere. On the other hand, the distrubance velocity $\underline{v}$ of the fluid assumes the value $-\underline{i}$ there. From equations (78) and (79) we see that this places the conditions

$$\frac{\partial \varphi}{\partial r} - \frac{1}{2k_1} \frac{\partial \chi_1}{\partial r} + \frac{1}{2k_2} \frac{\partial \chi_2}{\partial r} + (\chi_1 - \chi_2) \cos\theta = \frac{\partial \psi}{\partial r} ,$$

$$\frac{\partial \varphi}{\partial \theta} - \frac{1}{2k_1} \frac{\partial \chi_1}{\partial \theta} + \frac{1}{2k_2} \frac{\partial \chi_2}{\partial \theta} - (\chi_1 - \chi_2) \sin\theta = \frac{\partial \psi}{\partial r} ,$$

and

$$\frac{\partial \varphi}{\partial r} + \frac{2}{2k_1} \frac{\partial \chi_1}{\partial r} - \frac{\alpha_1}{2k_2} \frac{\partial \chi_2}{\partial r} - (\alpha_2 \chi_1 - \alpha_2 \chi_2) \cos\theta = -\cos\theta ,$$

$$\frac{\partial \varphi}{\partial \theta} + \frac{2}{2k_1} \frac{\partial \chi_1}{\partial \theta} - \frac{\alpha_1}{2k_2} \frac{\partial \chi_2}{\partial \theta} (\alpha_2 \chi_1 - \alpha_1 \chi_2) = -\sin\theta , \tag{81}$$

on the functions φ, χ_1, χ_2, and ψ at the sphere.

Now the general axially symmetric solutions of (68) and (77) outside the sphere are

$$\varphi = \sum_{n=o}^{\infty} An \frac{\partial^n}{\partial x^n} \left(\frac{1}{r}\right), \quad \chi_1 = e^{k_1 x} \sum_{n=o}^{\infty} B_n \frac{\partial^n}{\partial x^n} \left(\frac{e^{\mp k_1 r}}{r}\right) ,$$

$$\chi_2 = e^{k_x \chi} \sum_{n=o}^{\infty} C_n \frac{\partial^n}{\partial x^n} \left(\frac{e^{-k_2 r}}{r}\right) , \tag{82}$$

where the $\mp$ sign in χ_1, corresponds to $k_1 \gtrless o$ (because of the condition at infinity). The general axially symmetric solution of (80) inside the sphere is

$$\psi = \sum_{n=1}^{\infty} D_n \, r^{2n+1} \, \frac{\partial^n}{\partial x^n} \left(\frac{1}{r} \right) . \qquad (83)$$

When these are substituted into the four boundary conditions (81) and all functions are expanded in powers of k_1 and k_2, and then rearranged into spherical harmonics, we find that A_n, B_n, C_n and D_n satisfy certain algebraic relations. Since the spherical harmonics are independent, these algebraic relations provide an infinite set of equations for the coefficients A_n, B_n, C_n and D_n. However, we are only interested in the first-order effects in M, R and R_m. For this purpose the values of A_o, B_o and C_o suffice. They are

$$B_o = C_o = \frac{3}{2(\alpha_2 - \alpha_1)} \left\{ 1 + \frac{3(\pm\alpha_2 k_1 - \alpha_1 k_2)}{4(\alpha_2 - \alpha_1)} \right\} .$$

The bracket is correct to the first order in k_1 and k_2. The corresponding value of A_o is

$$A_o = - \frac{3(k_1 - k_2}{4(\alpha_2 - \alpha_1) k_1 k_2} \left\{ 1 + \frac{3(\pm\alpha_2 k_1 - \alpha_1 k_2)}{4(\alpha_2 - \alpha_1)} \right\} ,$$

$$= - \frac{3 R_m}{2(RR_m - M^2)} \left\{ 1 + \frac{3}{8} K \right\} ,$$

where

$$K = \begin{cases} R & \text{for } k_1 > 0. \\ \dfrac{2M^2 + R^2 - RR_m}{\sqrt{(R - R_m)^2 + 4M^2}} & \text{for } k_1 < 0. \end{cases}$$

The drag due to the Maxwell stresses is zero while the drag D due to pressure and viscous forces on the sphere can be calculated by using the formulas for the appropriate stresses as given in Lamb's book. After some calculations we get

$$D = D_o \left\{ 1 + \frac{3}{8} \chi R \right\} \tag{84}$$

where D_o is the value of the corresponding Stokes drag for nonconducting fluids and

$$\chi = \begin{cases} 1 & \text{for } M^2 < RR_m \\ \sqrt{\dfrac{R_m}{R}} \left\{ \dfrac{2M^2/RR_m + R/R_m - 1}{\sqrt{(\sqrt{R/R_m} - \sqrt{R_m/R})^2 + 4M^2/RR_m}} \right\} & \text{for } M^2 > RR_m \end{cases} \tag{85}$$

This shows that when the magnetic pressure $\mu\rho H_o^2$ is less than the dynamic pressure ρU^2 (i.e., $M^2/RR_m < 1$), the magnetic field has no influence on the drag. In the purely viscous case, $\sigma = 0$, $M = R_m = 0$ and $\chi = 1$, so that we recapture Oseen's result. On the other hand, when R and R_m are negligible, $\chi = M/R$ and we obtain the result of Stokes flow discussed in the previous section.

Chapter IV

Propagation of the Sound Waves

In this chapter we shall deal with a special case of the equations described in the last part. This specialization is achieved by ignoring the dissipative processes of viscosity, heat conduction, and electrical resistance. This leads to the following relatively simpler set of equations

$$\operatorname{div} \underline{H} = 0, \tag{1}$$

$$\frac{d\underline{H}}{dt} \quad \underline{H} \cdot \operatorname{grad} \underline{v} - \underline{H} \operatorname{div} \underline{v} = 0, \tag{2}$$

$$\rho \frac{d\underline{v}}{dt} + \operatorname{grad} p + \mu \underline{H} \times \operatorname{curl} \underline{H} = 0, \tag{3}$$

$$\frac{d\rho}{dt} + \rho \operatorname{div} \underline{v} = 0, \tag{4}$$

$$\frac{dS}{dt} = 0, \tag{5}$$

where we have omitted the subscript e from μ as there is no possibility of any confusion.

As pointed out in Chapter 1, Section 2, the equation (1) is a consequence of the equation (2). The rest of the equations form a system of eight non-linear partial differential equations of first order for the eight quantities $\underline{H}$, $\underline{v}$, ρ, and S. Furthermore, they belong to the same class of symmetric hyperbolic equations as do the equations of electrically non-conducting gas dynamics. The fact that these equations of magneto-hydrodynamics belong to a class of equations which are most consistent in their mathematical character supports the confidence in the physical

consistency of the various assumptions from which they were derived.

Alfven discovered transverse waves which propagate in incompressible inviscid fluids of negligible electric resistence at speed a, where

$$a^2 = \frac{\mu H_n^2}{\rho} \quad . \tag{6}$$

In a compressible, non-magnetic fluid for which $p = f(\rho, S)$, if viscosity and heat conductivity are negligible, the only possible waves are longitudinal waves travelling at the speed c, where

$$c^2 = \left(\frac{\partial p}{\partial \rho} \right)_S \tag{7}$$

In fluids which are both compressible and magnetic, it is natural to expect that waves of condensation in the velocity gradient will generally be neither transverse nor longitudinal, and that their speeds of propagation will differ from a and c. Then it follows from purely kinematical condiserations that not only spin, but also stretching and shearing of fluid elements are carried by such waves.

1. Kinematical preliminaries

We shall study the wave propagation from the point of view of singular surfaces. Sound waves are surfaces across which $\underline{H}$, $\underline{v}$, ρ, and p, are continuous, but some discontinuity is suffered by at least one of the partial derivatives of these quantities with respect to the spatial coordinates or the time. The surfaces which bear discontinuity in one or more of the quantities $\underline{H}$, $\underline{v}$, ρ, and p is called a shock wave and shall form the subject matter of the next Chapter. Wave fronts of the latter kind

were considered in Chapter 1, Section 4.

Thus, if F stands for any of the flow and field quantities, we have the relation

$$[F] = 0, \tag{8}$$

holding on the sonic discontinuity Σ. We suppose that the wave surface Σ has the equation $f(\underline{x}, t) = 0$, such that $\left| \text{ grad } f \right| > 0$. The normal vector $\underline{n}$ and the normal speed of advance G of Σ are (see the Appendix for the necessary details for understanding the relations 9-14)

$$\underline{n} = \frac{\text{gradf}}{\left|\text{gradf}\right|} , \tag{9}$$

$$G = - \frac{\frac{\partial f}{\partial t}}{\left|\text{grad } f\right|} . \tag{10}$$

Differentiate (8) along the surface, i.e.,

$$[F,_i] \; x^i_\alpha = 0, \tag{11}$$

in which α denotes partial differentiation with respect to the curvilinear coordinates u^α of the surface Σ. Now multiple both sides of the equation (11) by $g^{\alpha\beta} x^j_\beta$ and sum on the repeated indices. On account of the relation $g^{\alpha\beta} x^i_\alpha n^j_\beta = \delta^{ij} - n_i n_j$, this gives

$$[F,_i] \; (\delta^{ij} - n^i n^j) = o, \tag{12}$$

$$[F,_i] = n_j \left[\frac{dF}{dn} \right] . \tag{13}$$

or

$$[\text{grad } F] = \underline{n}\,\delta F, \quad \delta F = \left[\frac{dF}{dn}\right] . \tag{14}$$

The quantity δF is called the disturbance in F.

Let $\frac{\delta}{\delta t}$ denote the time derivatives as apparent to an observer moving with velocity $G\,\underline{n}$, viz.,

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + G\,\frac{dF}{dn} . \tag{15}$$

Observing that $\left[\frac{\delta F}{\delta t}\right] = \frac{\partial [F]}{\delta t}$, just as $[\underline{F} \cdot \underline{n}] = [\underline{F}] \cdot \underline{n}$, we have from (15):

$$\left[\frac{\partial F}{\partial t}\right] = - G \left[\frac{dF}{dn}\right] = - G\,\delta F \tag{16}$$

Combining (14) and (16) yields

$$\left[\frac{dF}{dt}\right] = \left[\frac{\partial F}{\partial t}\right] + \underline{v} \cdot [\text{grad } F] = - U\delta F, \tag{17}$$

where $U = G - \underline{v} \cdot \underline{n}$. With the help of (14), and (17), we write down

$$[\text{grad } \underline{v}] = \underline{n}\,\delta\,\underline{v}, \qquad \left[\frac{d\underline{v}}{dt}\right] = - U\,\delta\,\underline{v}, \tag{18}$$

$$[\text{grad } \rho] = \underline{n}\,\delta\,\rho \qquad \left[\frac{d\rho}{dt}\right] = - U\,\delta\,\rho, \tag{19}$$

$$[\text{grad } S] = \underline{n}\,\delta S, \qquad \left[\frac{dS}{dt}\right] = - U\,\delta S, \tag{20}$$

$$[\text{grad } p] = n\,\delta\,p = \left(\frac{\partial p}{\partial \rho}\right)_S [\text{grad } \rho] + \left(\frac{\partial p}{\partial S}\right)_\rho [\text{grad } S]$$

$$= \underline{n}\,c^2\,\delta\,\rho + \underline{n}\,B\,\delta\,S, \tag{21}$$

$$[\text{grad } \underline{H}] = n\,\delta\,\underline{H}, \qquad \left[\frac{d\underline{H}}{dt}\right] = - U\,\delta\,\underline{H}, \tag{22}$$

where $B = \left(\frac{\partial p}{\partial S}\right)_{\rho}$.

Let us further observe from (14) that

$$[\operatorname{div} \underline{F}] = \underline{n} \cdot \delta \underline{F}, \quad [\operatorname{curl} \underline{F}] = \underline{n} \times \delta \underline{F} \quad . \tag{23}$$

2. Friedrichs - Kranzer equation for normal speeds

Since the equations (2-6) hold on both sides of the sonic discontinuity we find by taking the jumps and using the relations (18) to (22), the results

$$\begin{aligned}
&- U \delta \underline{H} - H_n \delta \underline{v} + H (\delta \underline{v}) \cdot \underline{n} = 0, \\
&- \rho U \delta \underline{v} + c^2 \underline{n} \delta \rho + B \underline{n} \delta S + \mu \underline{n} (\underline{H} \cdot \delta \underline{H}) - \mu H_n \delta \underline{H} = 0, \\
&- U \delta \rho + (\delta \underline{v}) \cdot \underline{n} = 0, \\
&- U \delta S = 0.
\end{aligned} \tag{24}$$

where we have used the fact that $\underline{H} \times \operatorname{curl} \underline{H} = -\operatorname{grad} \frac{H^2}{2} + (\underline{H} \cdot \operatorname{grad}) \underline{H}$. This is a definite system of eight equations in eight quantities $\delta \underline{v}$, $\delta \underline{H}$, $\delta \rho$ and δS. The determinant of this system of homogeneous equations is

$$\rho U^2 (\rho U^2 - \mu H_n^2) \left\{ \rho U^4 - (\rho c^2 + \mu H^2) U^2 + c^2 \mu H_n^2 \right\}, \tag{25}$$

where $H_n = \underline{H} \cdot \underline{n}$. If we put this determinant equal to zero, we find the characteristic velocities of the waves. All the eight roots of this equation are seen to be real in accordance with the fact that the differential equations are hyperbolic. Observe that if the equation $\operatorname{div} \underline{H} = 0$ is also taken into account, then the first factor is merely ρU and the system became seventh order.

Let us first discuss the roots of the last factor of the determinant (25). The conditions that this factor vanishes can be written in the form

$$U^2 \ (\rho U^2 - \mu H^2) = c^2 \ (\rho U^2 - \mu H_n^2), \tag{26}$$

or in the form

$$(U^2 - c^2)(U^2 - \frac{\mu H_n^2}{\rho}) = \frac{U^2 \mu(H^2 - H_n^2)}{\rho} \ , \tag{27}$$

The larger and smaller of the roots $U > 0$ of this equation will be denoted, respectively, by U_{fast} and U_{slow}. Since the right hand side of (27) is non-negative we deduce the inequalities

$$U_{slow} \leq c \leq U_{fast} \ , \tag{28}$$

and

$$U_{slow} \leq a \leq U_{fast} \ , \tag{29}$$

where a, the Alfven speed, is defined by the relation (6). The sound speed c is as given by the relation (7). An equality sign can hold in relations (28) and (29) only $\underline{H} = H_n \underline{n}$, so that the right member of (27) vanishes. In this case one of the two speeds equals c, the other equals a.

The Alfven wave, also called as the intermediate wave, belongs to the factor, $(U^2 - \mu H_n^2/\rho)$ of the determinant (25). As seen from (29), the speed a lies between U_{slow} and U_{fast} except if $\underline{H}$ is parallel to $\underline{n}$ or normal to $\underline{n}$.

From $(24)_4$ we observe that either $U = 0$, $\delta S \neq 0$, or $U \neq 0$, $\delta S = 0$. Let us first dispose of the case

$$dS \neq 0.$$

The wave front in this case is a material surface (contact discontinuity). This corresponds to the first factor of the equation (25) i.e., U^2. This root has the multiplicity 2 unless $H_n = 0$, in which case the multiplicity is 6. When $H_n \neq 0$, the relations (24) possess only the solutions.

$$\begin{aligned} \delta\underline{H} &= k\underline{n}, \\ \delta\underline{v} &= 0, \qquad \delta\rho = 0, \\ \delta S &= k_1 , \end{aligned} \tag{30}$$

with arbitrary k and k_1. Because of equation (1) $\delta H_n = 0$, therefore $k = 0$. From $(30)_2$ we observe that in striking contrast to the situation in gas-dynamics: a material surface in a magnetized fluid does not permit a discontinuity in the tangential component of the velocity, provided $H_n \neq 0$.

In the rest of the Chapter we discuss the important case: $\delta S = 0$.

3. Classification of different Modes of Wave Propagation

Let us split a vector $\underline{F}$ into its normal and tangential components, viz.,

$$\begin{aligned} \underline{F} &= F_n \underline{n} + \underline{F}_T = (\underline{F} \cdot \underline{n})\underline{n} - \underline{n} \times \underline{F}_{\vdash} , \\ \underline{F} &= \underline{n} \times \underline{F}_T . \end{aligned} \tag{31}$$

Thus $F_{\vdash}$ is the vector obtained by rotating the tangential component of F_T of F through 90 degrees counter-clockwise about the normal $\underline{n}$. For example

$$\begin{aligned} \underline{H} &= H_n \underline{n} + \underline{H}_T = (\underline{H} \cdot \underline{n})\underline{n} - \underline{n} \times H_{\vdash} . \\ \delta\underline{F} &= \delta\underline{F}_n + \delta\underline{F}_T . \end{aligned} \tag{32}$$

From the relations (1) and (23), it follows that $\delta \underline{H}_n = 0$. Let us denote the transversal vector $\delta \underline{H}_T$ by $\underline{\beta}$. Similarly $\delta \underline{v} = \delta v_n + \delta \underline{v}_T$. From relations $(24)_3$, it follows that $\delta v_n = U \frac{\delta \rho}{\rho}$ or $\delta v_n = U \delta(\log \rho)$. If we define the condensation ε of the wave by $\delta \log \rho$, then $\delta v_n = U\varepsilon$. Furthermore, let us denote $\delta \underline{v}_T$ by $\underline{\alpha}$. Thus instead of the seven quantities $\delta \underline{v}$, $\delta \underline{H}$, and $\delta \rho$, we now have equal number of quantities $\underline{\alpha}$, $\underline{\beta}$ and ε. Both $\underline{\alpha}$ and $\underline{\beta}$ are tangent to the wave front. To sum up we have put

$$\delta \underline{H} = \underline{\beta} \ , \quad \delta \underline{v} = \underline{\alpha} + U \varepsilon \ \underline{n}, \quad \delta \rho = \rho \varepsilon \ . \tag{33}$$

Also for the vorticity $w = \operatorname{curl} \underline{v}$, and electric current density $\underline{J}$, we have

$$[\underline{w}] = \underline{n} \times \underline{\alpha} \ , \quad \underline{\alpha} = -\underline{n} \times [\underline{w}] \tag{34}$$

and

$$[\underline{J}] = \underline{n} \times \underline{\beta} \ , \quad \underline{\beta} = -\underline{n} \times [\underline{J}] \ . \tag{35}$$

In the sequel we shall come across the terms current sheet and vortex sheet. A vortex sheet is a surface across which the vorticity is discontinuous, but the density is continuous; and there is a similar definition for the current sheet. We have already mentioned the term material surface in the previous section. 'Material' means bound to the surface.

Now the equations (34) take the following form for

(1) Compressible fluids

$$- U\underline{\beta} + (\underline{H}_n \ \underline{n} + \underline{H}_T) \ U \varepsilon - H_n(\underline{\alpha} + U\varepsilon \underline{n}) = 0, \tag{36}$$

$$- U \ (\underline{\alpha} + U\varepsilon \underline{n}) + \rho c^2 \varepsilon \ \underline{n} - \mu \ (\underline{n} \times \underline{\beta}) \times \underline{H} = 0 \ . \tag{37}$$

Since $-(\underline{n} \times \underline{\beta}) \times \underline{H} = (\underline{\beta} \cdot \underline{H}_T)\,\underline{n} - H_n \underline{\beta}$ the relation (36) and the tangential component of (37) become

$$-H_n \underline{\alpha} - U\underline{\beta} + U\varepsilon \underline{H}_T = 0, \tag{38}$$

$$-\rho U \underline{\alpha} - \mu H_n \underline{\beta} = 0. \tag{39}$$

It is easier to interpret these equations if we rotate all vectors through a right angle. To the cross product of (38) and (39) by $\underline{n}$, simplified by use of (32), (34) and (35), we subjoin the normal component of (37), thus obtaining the definitive system:

$$\rho U [\underline{w}] + \mu H_n [\underline{J}] = 0, \tag{40}$$

$$H_n [\underline{w}] + U [\underline{J}] = -\varepsilon U \underline{H}\,\vdash \tag{41}$$

$$\rho \varepsilon (U^2 - c^2) = -\mu [J]\, \underline{H}\,\vdash, \tag{42}$$

valid for all singular surfaces in compressible fluids.

(2) Incompressible fluids

In this case the equations (24) reduce to

$$-U\underline{\beta} - H_n \underline{\alpha} = 0, \tag{43}$$

$$-\rho U \underline{\alpha} + \delta p \underline{n} - \mu (\underline{n} \times \underline{\beta}) \times \underline{H} = 0. \tag{44}$$

By precisely the same analysis as given for compressible fluids, the equations (43) and (44) lead to

$$\rho U [\underline{w}] = -\mu H_n [\underline{J}], \tag{45}$$

$$H_n [\underline{w}] = -U [\underline{J}], \tag{46}$$

$$\delta p = \mu [\underline{J}] \cdot \underline{H}\,\vdash \tag{47}$$

Before we give the rest of the analysis, let us recapitulate the mode of sound propagation in gas-dynamics. This is embodied in the Hugonoit-Hadamard theorem: "In a compressible non-magnetic fluid, only the following kinds of singular surfaces are possible:

(1) Material vortex-sheets

(2) Longitudinal waves propagating at a speed in any direction. The first statement in the Hugonoit-Hadamard theorem asserts that transversal discontinuities cannot propagate, but must eternally divide one portion of fluid from another.

The corresponding theorem for Alfven waves is: In an incompressible magnetic fluid only the following kinds of singular surfaces are possible:

1. Material vortex-current sheets. These must be everywhere tangent to the lines of induction (lines of magnetic force); the jumps of vorticity and current density are tangent to the surface but otherwise arbitrary.

2. Transverse waves carrying jumps of current density and vorticity satisfy the relations (45) to (47).

From (45) and (46) we see at once that if $H_n = 0$, i.e., if the wave front is tangent to $\underline{H}$, then $U = 0$, and we fall back on the case 1. This kind of a singular surface again permanently divides two portions of fluids, but it must be a surface swept out by lines of induction. The jumps of $\underline{w}$ and $\underline{J}$ are arbitrary, so long as they are tangential. If $H_n = 0$, it follows from (45) and (46) that $U^2 = a^2 = \mu H_n^2 / \rho$. Thus <u>any wave front not tangent to the magnetic field is propagated at Alfven speed</u> $\pm a$. Note

that a^2, unlike c^2, depends upon the orientation of the wave front relative to the magnetic field, being greatest for a wave which moves straight down the lines of induction; then $H_{\vdash} = 0$, and hence $\delta p = 0$, also. Formally, Case 1 may be subsumed under Case 2, since $a = 0$, if $H_n = 0$. In both cases, (45) holds, but in the former, it is vacuus, while in the latter, it gives the connection between vorticity and electric current density: $[\underline{w}]$ and $[\underline{J}]$ point in opposite directions if $UH_n > 0$, in the same direction if $U H_n < 0$, while their magnitudes stand in the ratio $\mu \left|H_n\right| / \rho U$.

Compressible Fluids

The relation (40) is the same as (45), it is the universal relation between spin and electric current density. The interpretation we have given for it in the case of an incompressible fluid was so worded as to remain valid for compressible fluids also. However, the variety of possible singular surfaces is greater. In order to classify them we assume that $\rho > 0$, $c^2 > 0$, $H \neq 0$, and we shall tacitly discard cases when the only solutions are $\varepsilon = 0$, $[\underline{w}] = 0$, $[\underline{J}] = 0$, i.e., when all quantities are actually continuous.

Case 1. $\varepsilon = 0$: The surface is a vortex sheet. Then by (42), $[\underline{J}]$ is parallel to $\underline{H}_T$ if $\underline{H}_T \neq 0$.

Case 1a. If $H_n \neq 0$, $\varepsilon = 0$, then $[\underline{w}]$ is also parallel to $\underline{H}_T$ if $H_T \neq 0$, and $U^2 = a^2$. If $\underline{H}_T = 0$, $[\underline{J}]$ may be arbitrary, but again $U^2 = a^2$. Thus there are propagating vortex-sheets, purely transversal waves which propagate at Alfven speed, carrying jumps of vorticity and current density connected by (40). If the wave is moving

straight down the lines of induction, the jump in current density may be arbitrary both in magnitude and in direction; if the wave is inclined to the magnetic field but not normal to it, the magnitude of the jump of current density may be arbitrary, but its direction must be that of $\underline{n} \times \underline{H}$.

Case 1b. If $H_n = 0$, $\varepsilon = 0$, the system (40-42) becomes $\rho U [\underline{w}] = 0$, $U [J] = 0$, $[\underline{J}] H_{\vdash} \neq 0$. Therefore, since $H_{\vdash} \neq 0$, $[\underline{J}]$ is parallel to $\underline{H}$, but $U = 0$, and $[\underline{w}]$ is arbitrary. Thus a vortex-sheet tangent to the lines of induction is necessarily material, it can carry any jump of vorticity; the jump of current density must be parallel to $\underline{H}$ but is arbitrary in magnitude. Since $a = 0$ when $H_n = 0$, this case may be regarded as a limiting case of the former one. (The greater variety of material vortex-sheets are possible if the fluid is incompressible; it follows from the fact that δp is then restricted only by (47), while in a compressible fluid $\delta p = \rho \varepsilon c^2$, so that $\delta p = 0$ if $\varepsilon = 0$).

Case 2: $\varepsilon \neq 0$: The surface is a sonic disturbance. If $H_{\vdash} = 0$, from (42) we now get $U^2 = c^2$, while from (40) and (41) we see that either $U = 0$ or $U^2 = a^2$. This is a contradiction unless $a^2 = c^2$. Hence, there follows a major difference between the compressible and incompressible cases: No sonic wave can propagate straight down the lines of induction except when $a^2 = c^2$. That is $(\partial p/\partial \rho)_S = \mu H^2/\rho$. For an ideal gas, this relation reduces to $\gamma p = \mu H^2$. Such a delicate balance between the static and magnetic pressure would be extraordinary indeed, and henceforth we exclude it.

With the help of the relation of (31), the equation (27) becomes

$$(U^2 - c^2)(U^2 - a^2) = U^2 \frac{\mu H_T^2}{\rho} . \qquad (48)$$

Case 2a. $U = 0$, $\varepsilon = 0$: The singularity is a material condensation. Then from (48) we see that $a^2 = 0$; therefore $H_n = 0$. The system (40-42) reduces to one member, viz.,

$$\rho \varepsilon c^2 = \mu [\underline{J}] \cdot \underline{H} . \qquad (49)$$

This case includes Case 1b in the limit when $\varepsilon \to 0$, and it includes the limit case of Case 1a obtained when $a \to 0$. For arbitrary ε, summary of Cases 1b and 2a shows that every material singular surface is tangent to the lines of induction; the most general such surface may be decomposed into two:

(i) A vortex-current sheet carrying arbitrary jumps in $\underline{w}$ and $\underline{J}$.

(ii) A surface carrying a longitudinal condensation ε obtained by (49).

That a jump of current density which is inclined to the magnetic field gives rise to a material current-sheet which is the seat of a condensation is perhaps the most remarkable effect of the interaction between compressibility and magnetism. If the jump of current density lies in the half-plane to the left of $\underline{H}$, the condensation is positive; if in the right plane, negative. Thus a suitable tangential flow of current can render in a sheet of induction a material boundary permanently separating a region of dense gas from a region of rare gas.

Case 2b. $H_n = 0$, $U \neq 0$, $\varepsilon \neq 0$: The surface is a propagating

condensation always tangent to the lines of induction. Since a = 0, (48) yields

$$U^2 = c^2 + \frac{\mu H_T^2}{\rho} = c^2 + \frac{\mu H^2}{\rho}, \tag{50}$$

and the system (40-42) reduces to the form

$$[\underline{w}] = 0, \quad [\underline{J}] = -\varepsilon \underline{H}_{\vdash}, \quad \varepsilon H^2 = -[\underline{J}] \cdot \underline{H}_{\vdash} \tag{51}$$

so that these waves are irrotational, they carry a jump of current density which is of magnitude $|\varepsilon H|$ and is in the direction opposite to $\varepsilon \underline{n} \times \underline{H}$; they travel at supersonic speed. As $H \to 0$, these waves pass continuously into ordinary waves of sound.

The cases considered so far are special and in some way degenerate. We come now to the general case when the universal relation (40) assumes its full powers.

Case 2c. $H_n \neq 0$, $U \neq 0$, $\varepsilon \neq 0$. We see from (40) that now neither $[\underline{J}]$ nor $[\underline{w}]$ may vanish, and that these vectors are parallel to each other by (41) each is parallel to $\underline{H}_{\vdash}$. The general solution of the system (40-48) may be written in terms of a scalar parameter δ:

$$[\underline{J}] = \delta (\underline{n} \times \underline{H}), \tag{52}$$

$$[\underline{w}] = -\frac{\mu H n \delta}{eU} (\underline{n} \times \underline{H}), \tag{53}$$

$$\varepsilon = \left(\frac{a^2}{U^2} - 1 \right) \delta. \tag{54}$$

Let us think of the condensation ε and the magnetic field $\underline{H}$ as given. $\underline{H}$ and the wave $\underline{n}$ determine two pairs of roots U^2_{fast} and U^2_{slow} as given by the relations (28) and (29). However the signs of equality

of the relations (28) and (29) are not possible in the present case for the following reason. In fact $w^2 = a^2 \neq 0$ yields $H^2_T = 0$ by (48); hence by (42), since $\varepsilon = 0$, it follows that $U^2 = c^2$. Likewise, $U^2 = c^2$ implies that $U^2 = a^2$. Thus $c^2 = a^2$, but this is the case, excluded here, when (41) holds. Therefore, $U^2_{fast} > a^2$ and $U^2_{slow} < a^2$; also $U^2_{fast} > c^2$ and $U^2_{slow} < c^2$. By (54), then, δ has the same sign as ε for the slow waves but the opposite sign for the fast waves. By (52) to (54) we see that for given values of H_n, $\underline{n} \times \underline{H}$, and ε there are exactly four possible waves. Summary of the implication of these equations yields the main theorem: Aside from the special circumstances in which (41) holds, no wave carrying a condensation can propagate down the lines of induction, but any other direction of propagation is possible. In the general case, when the wave front is not tangent to the lines of induction, there are four possible waves, determined uniquely by $\underline{H}$, $\underline{n}$ and ε. One pair travels at supersonic speed, the other at subsonic. The jumps of current density and vorticity are parallel to $\underline{n} \times \underline{H}$. The sense of $[\underline{J}]$ is the same as that of $\underline{n} \times \underline{H}$ for a slow positive or fast negative condensation, the opposite in the opposite cases. Both members of one pair carry the same condensation and the same jump of current density; they differ only in the sign of U, which is either + or - at will, and in the sense of $\underline{w}$, which is opposite to that of $\underline{J}$ or the same, according as $U\,\underline{n} \cdot \underline{H} > 0$ or $U\,\underline{n} \cdot \underline{H} < 0$. In every case, the magnitudes of all jumps stand in the ratio

$$\left| [\underline{J}] \right| : \left| [\underline{w}] \right| : \left| \varepsilon \right| = \left| \underline{H}_T \right| : \frac{\mu \left| H_n \underline{H}_T \right|}{\rho |U|} : \left| \frac{\mu H_n^2}{\rho u^2} - 1 \right| . \quad (55)$$

As $H \to 0$, the fast waves go over into ordinary waves of sound, longitudinal waves carrying an arbitrary condensation in an arbitrary direction and propagate at speed $\pm c$, while the slow waves become indeterminate material singularities. As $\varepsilon \to 0$, both kinds of waves go over into Alfven waves.

4. Fluid and magnetic distortion carried by the wave front

In view of the relations (33), the equations (18) and $(22)_1$, can be written as

$$[\operatorname{grad} \underline{v}] = \underline{\alpha}\, \underline{n} + [\operatorname{div} \underline{v}]\ \underline{n}\ \underline{n},$$

and

$$[\operatorname{grad} \underline{H}] = \underline{\beta}\ \underline{n}.$$

Substituting the value β in terms of α as obtained from (39), we readily derive the result:

$$\rho \underline{U} \left\{ [(\operatorname{grad} \underline{v})] - [(\operatorname{div} v)]\ \underline{n}\ \underline{n} \right\} + \mu H_n\ [\operatorname{grad} \underline{H}] = 0. \quad (56)$$

The equation (40) is the skew-symmetric part of this equation. To interpret the symmetric part of the equation (56), we set

$$\underline{\underline{e}} = \frac{1}{2} \operatorname{grad} \underline{v} + \frac{1}{2} (\operatorname{grad} \underline{v})^T - (\operatorname{div} \underline{v})\ \underline{n}\ \underline{n},$$

$$\underline{\underline{\gamma}} = \frac{1}{2} \operatorname{grad} \underline{H} + \frac{1}{2} (\operatorname{grad} \underline{H})^T,$$

where the transcript T denotes transposition. If we imagine a small mass of equally charged particles suddenly released in the fluid, in the first instant they will acquire velocities proportional to $\underline{H}$, accordingly,

γ measures the initial stretchings and shearings of lines of such charges. Thus $\underline{\underline{\gamma}}$ may be called the magnetic distortion. The tensor $\underline{\underline{e}}$ is a measure of fluid distortion. Namely, the component e_{nn} is the mean rate of shrinking, per unit area, of the surface of the wave front: $e_{n'n'}$, if n' lies in the wave front, is the stretching of n', and $-\frac{1}{2}\ e_{n'n''}$ is the shearing of the pair in the perpendicular directions n' and n'', presumed to be either normal to the wave front or parallel to it. Taking the symmetric part of (56) yields

$$e\ U\ [\underline{\underline{e}}] \quad + \quad \mu\ H_n\ [\underline{\underline{\gamma}}] \quad = \quad 0 \tag{57}$$

a relation of just the same form as (40). Therefore, the disturbances of magnetic distortion and fluid distortion carried by a magnetosonic wave are related in just the same way as are the disturbances of fluid spin and electric current density. In the previous section we have classified the singular surfaces according to speeds of propagation and the disturbances carried by the surfaces. The foregoing theorem renders superfluous any further classification when it is desired to specify the nature of accompanying fluid and magnetic distortions. All that is needed is to replace $\underline{w}$ by $\underline{\underline{e}}$ and $\underline{j}$ by $\underline{\underline{\gamma}}$ in the results we have already discussed.

5. Geometric shape of wave fronts

To construct the surfaces of normal speed, we go back to the equation (25) which gives the various normal speeds of advance of the magnetohydrodynamic wave fronts. Since the dimensions of the quantity $(\mu/\rho)^{\frac{1}{2}}\ H$, are those of the velocity, the quantity $(U^2\rho/\mu\ H^2)$, is dimensionless. Let us define it as

$$r^2 = U^2 \rho / \mu H^2. \tag{58}$$

Also define a parameter b as

$$b^2 = c^2 \rho / \mu H^2, \tag{59}$$

and let θ denote the angle between $\underline{n}$ and $\underline{H}$. With this notation the non-zero roots of the equation (25) are

$$r_{slow} = \left\{ \tfrac{1}{2} (1 + b) - \tfrac{1}{2} \left\{ (1 + b)^2 - 4b \cos^2\theta \right\}^{\frac{1}{2}} \right\}, \tag{60}$$

$$r_{alfven} = \cos \theta, \tag{61}$$

$$r_{fast} = \left\{ \tfrac{1}{2} (1 + b) + \tfrac{1}{2} (1 + b)^2 - 4b \cos^2 \theta)^{\frac{1}{2}} \right\} \tag{62}$$

Figure 4 is a plot, on polar coordinates paper, of r against θ for

(a) $b < 1$

(b) $b = 1$

(c) $b > 1$

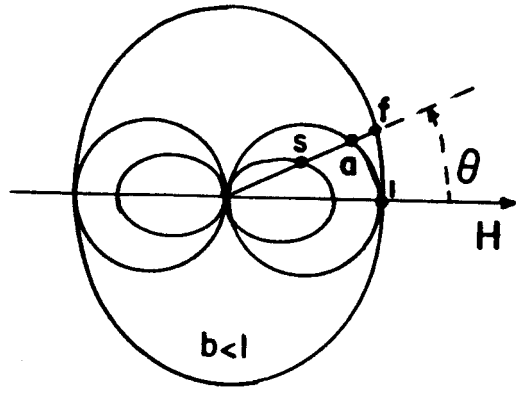
By rotating each figure about the $\underline{H}$-axis, we obtain a surface in three space which is called the surface of normal speeds.

The following observations may be made at once by looking at figure 1. The intermediate locus as given by (61), consists of two spheres through the origin. The slow locus also has two loopes meeting at the origin, while the fast locus is a simple smooth surface centered about the origin. The transverse locus separates the slow locus from the fast one. In any given direction $\underline{n}$, a slow wave will propagate more slowly than the intermediate wave (Alfven wave) in the same direction, and this, in turn will be slower than the corresponding fast plane wave. For $b < 1$, the intermediate locus touches the fast locus (when n is parallel to $\underline{H}$) while for $b > 1$ it touches the slow locus. For $b = 1$ and $\underline{n}$

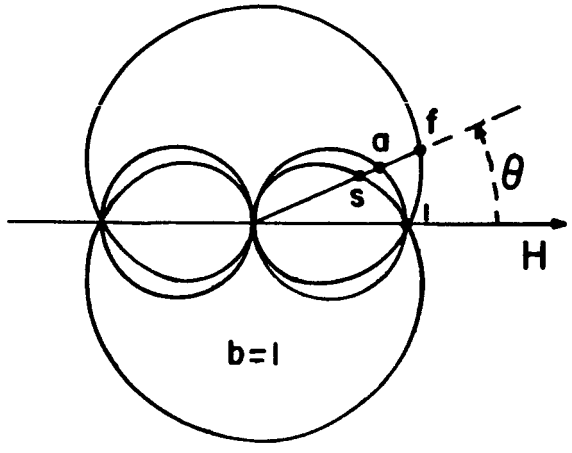
parallel to $\underline{H}$, all the three loci have common points.

These figures have, however, not brought out clearly the anistropy of the wave fronts. To plot the wave fronts it is necessary to recall that the length of the vector from the origin to a point on the curve in figure 1, represents the distance travelled by the plane wave normal to that vector, and the wave front due to an isotropic pulse can be obtained by finding the envelope of these normal planes. They are obtained by rotating the curves in figure 2, about the $\underline{H}$-axis. The surfaces thus obtained are also called the ay surfaces for the following reason. Each factor of the equation (25), when put equal to zero, is itself a partial differential equation of first order, because $\underline{n}$, the unit normal vector is equal to (grad f) $\Big|$ grad f $\Big|$. As such the surfaces f (x, t) given by this first order differential equation can be spanned by its characteristic curves, known as bicharacteristics or rays for the present case.

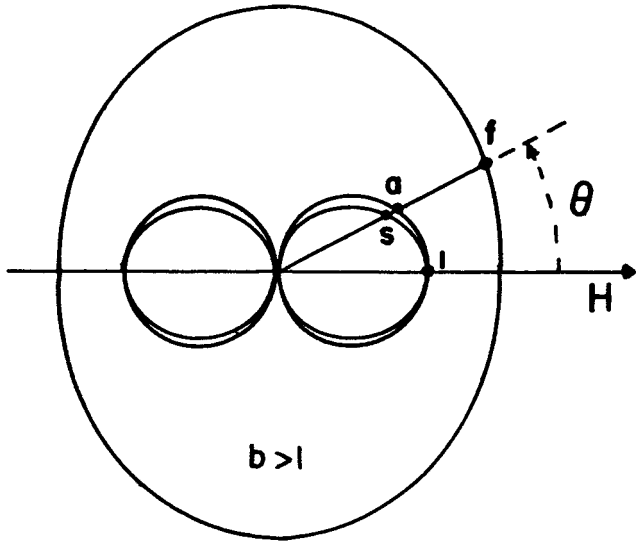
From figure 2, we observe that the outer fast locus is somewhat anistropic but resembles the sonic wave front. The locus for the intermediate (or Alfven) wave is extremely degenerate and consists of two points on the $\underline{H}$-axis together with the included line segment. This degenerate locus represents strictly one-dimensional propagation along magnetic lines. Thus the Alfven wave front is forever contained within a fixed tube. The slow locus is even more singular. It consists of two disconnected cusped regions together with the intercepted line segment. The qualitative significance is easily appreciated by the example in Figure 3. This figure depicts the evolution of an originally



(a)

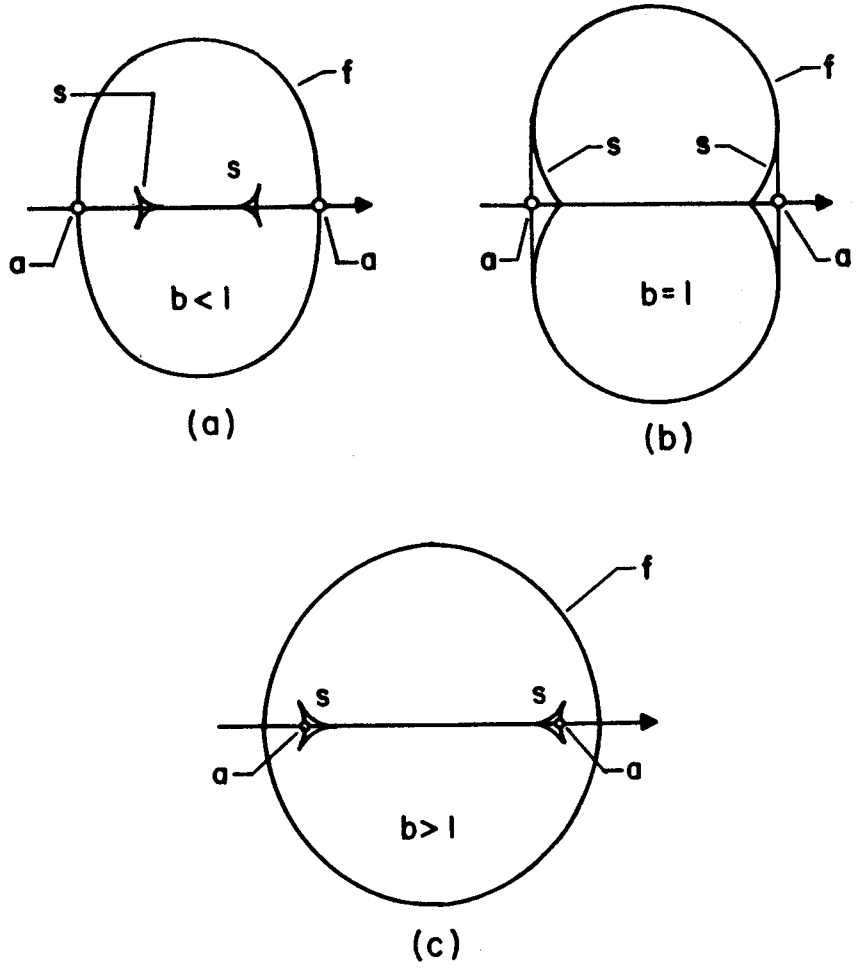


(b)



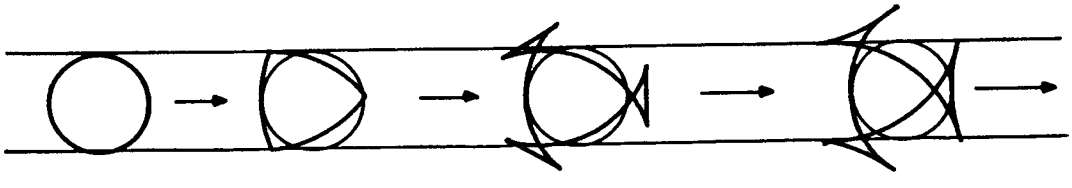
(c)

FIGURE 1



RAY SURFACE

FIGURE 2



IT ILLUSTRATES THE EVOLUTION OF A SLOW WAVE FRONT FROM A SPHERICAL INITIAL MANIFOLD

FIGURE 3

spherical slow wave-front in a uniform medium. It overtakes the Alfven wave after singularities develop in the wave front. Summarizing, the intermediate wave propagates one-dimensionally along flux-tubes, the slow wave front is confined to a certain cone surrounding the magnetic field direction (it is displaced along the magnetic lines of force), and only the fast wave front propagates in all directions slightly anisotropically.

Chapter V

Shock Waves

Non-dissipative gas dynamical motions, involving a compression, are known to break down at some time. From that time on shock discontinuities appear. The flow and field quantities on both sides of the shock front are governed by the laws of conservation. The same phenomenon must be expected to happen in hydromagnetic motions if the compressibility of the fluid is taken into account. In Chapter 1 we considered this point in full generality. If in the difference equations (33-36) of that chapter we neglect the terms involving viscosity, heat conductivity and electrical resistance, we find:

$$[H_n] = 0, \tag{1}$$

$$[\rho (v_n - G)] = 0, \tag{2}$$

$$[(v_n - G)\,\underline{H} - H_n\,\underline{v}] = 0, \tag{3}$$

$$[\rho (v_n - G)\,\underline{v} + (p + \mu \frac{H^2}{2})\,\underline{n} - \mu H_n \underline{H}] = 0, \tag{4}$$

$$(v_n - G)(\frac{1}{2}\rho v^2 + \rho E + \mu \frac{H^2}{2}) + v_n (p + \mu \frac{H^2}{2}) - \mu H_n (\underline{H} \cdot \underline{v}) = 0. \tag{5}$$

We assume the direction of the normal vector $\underline{n}$ so chosen that fluid crosses the shock front in the direction of the normal, i.e., $(v_n - G) \geq 0$. If we are given the values of $\underline{H}_1$, $\underline{v}_1$, ρ_1, and p_1; we can readily calculate the values of the corresponding quantities on side 2 of the shock by the help of the above shock conditions. In fact if we define the shock strength as

$$\delta = [\rho] / \rho_1, \tag{6}$$

the required results are:

$$[\underline{v}] = \frac{\delta\,(v_{1n} - G)\,\mu\,H_{1n}}{\rho_1\,(v_{1n} - G)^2 - (1+\delta)\,\mu\,H_{1n}^2}\,\underline{H}_{1T} - \frac{\delta}{1+\delta}\,(v_{1n} - G)\,n, \quad (7)$$

$$[\underline{H}] = \frac{\delta\ 1\ (v_{1n} - G)^2}{\rho_1\,(v_{1n} - G)^2 - (1+\delta)\,\mu\,H_{1n}^2}\,\underline{H}_{1T}, \quad (8)$$

$$[p^*] = \frac{\delta}{1+\delta}\ \rho_1\ (v_{1n} - G)^2\ . \quad (9)$$

The subscripts n and T stand for, as in the previous chapters, the normal and tangential components of the quantities involved. Since $E = E(p, \rho)$, the value of δ is effectively calculated from the shock condition (5).

For classifying the various shock discontinuities, we shall write the shock conditions in a slightly different form. In that connection we set

$$\tau = \frac{1}{\rho}\ , \quad (10)$$

$$m = \rho\ (v_n - G)\ . \quad (11)$$

Since m and H_n are continuous across a shock wave, the remaining difference equations (3-5) can be written as

$$m\left[E + \frac{p}{\rho} + \frac{1}{2}\,m\,\tau^2 + \frac{1}{2}\,\underline{v}_T^2 + \tau\,\mu\,\underline{H}_T^2\right] = \mu\,H_n\,[\underline{H}_T - \underline{v}_T]\ , \quad (12)$$

$$[p] + m^2\,[\tau] + \frac{1}{2}\,\mu\,[\underline{H}_T^2] = 0, \quad (13)$$

$$m\,[\underline{v}_T] = \mu\, H_n\, [\underline{H}_T] \quad , \tag{14}$$

$$H_n\, [\underline{v}_T] = m\, [\,\tau\, \underline{H}_T] \,. \tag{15}$$

1. Various speeds of shock propagation

In the last chapter we found that a sonic disturbance travels at various speeds. It is natural to pose the same question about the propagation of shock waves. To answer that question we use the notation of mean value of a quantity, i.e., $\widetilde{Q}$ as

$$\widetilde{Q} = \frac{1}{2}\,(Q_1 + Q_2)\,, \tag{16}$$

where, as before, the subscripts 1 and 2 stand for the two sides of the shock front. For a product we shall have

$$[PQ] = \widetilde{P}\,[Q] + [P]\,\widetilde{Q}. \tag{17}$$

In this notation the shock relations (2-5) can be written as

$$m\,[\tau] - [v_n] = 0, \tag{18}$$

$$m\widetilde{\tau}\,[\underline{H}] + \widetilde{H}\,[v_n] - \underline{H}_n\,[\underline{v} - G\,\underline{n}] = 0 \tag{19}$$

$$m\,[\underline{v} - G\,\underline{n}] + [p]\;\underline{n} + \mu\widetilde{\underline{H}}\,[\underline{H}] - M\,H_n\,[\underline{H}] = 0 \,. \tag{20}$$

This is a determinate system of seven equations in seven quantities $[\underline{H}]$, $[\underline{v}]$, $[\tau]$, (note that $[p]$ and $[\tau]$ are related). Since a shock front has to bear a discontinuity in at least one of these seven quantities, the discriminant of this algebraic system must be zero. This leads to the Friedrichs-Kranzer equation

$$\widetilde{\tau}^2 m\,(\widetilde{\tau}\,m^2 - \mu\,H_n^2)\,(\widetilde{\tau}\,m^4 + (\widetilde{\tau}[\tau]^{-1})\,[p] - \mu\widetilde{H}^2)\,m^2 - [\tau]^{-1}\,[p]\,\mu\,H_n^2 = 0 \tag{21}$$

This is an equation for the flux m, but it may be just as well considered as an equation for the shock velocity

$$G = \tilde{v}_n - m\tilde{\tau} \tag{22}$$

It is clear from (21) that there are slow, intermediate, and fast shocks and that the relationship between them is the same as that between the corresponding sound waves.

2. Classification of Shock Fronts

As in the previous chapter, let us first discuss the degenerate cases.

Case 1: $m = 0$, $[\tau] \neq 0$. Since there is no flow across the front, the surface if a material discontinuity.

Case 1a: $H_n \neq 0$. We observe from the equations (12) to (15) that

$$[\underline{v}_T] = 0, \quad [p] = 0, \quad [\underline{H}_T] = 0. \tag{23}$$

Thus the velocity, pressure and magnetic field are continuous. The jumps in density (hence in entropy and temperature, etc.) are arbitrary.

Case 1b: $H_n = 0$. If we put $m = 0$, $H_n = 0$, the equations (12), (14), and (15) are identically satisfied. At such a shock the jumps $[\underline{v}_T]$ and $[\underline{H}_T]$ are arbitrary. Thus the velocity and magnetic field are transversal and can have any discontinuity in both magnitude and direction. The jump in total pressure is zero, i.e., $[p^*] = 0$ (this follows from the relation (13)). The density discontinuity can, however, take any value. This type of shock front is called a tangential discontinuity or a shear flow discontinuity. It can be proved that in a magnetized fluid no transversal shock can be maintained if the magnetic field has a normal

component $H_n \neq 0$, at the front. If such a discontinuity is present at an initial time, a wave motion must result which resolves it.

Case 2: $[\tau] = 0$, $m \neq 0$. Since the flux $m = (v_n - G)/\tau$ is continuous, the normal velocity component is continuous also, i.e., $[v_n] = 0$. On the right-hand side of (15) we can take τ outside the brackets and divide this equation by (14), getting thereby

$$m = \sqrt{\frac{\mu}{\tau}} \quad H_n, \tag{24}$$

and

$$[\underline{v}_T] = \sqrt{\mu \tau} \quad [\underline{H}_T] . \tag{25}$$

The equation (12) then becomes

$$m[E] + m\tau[p + \mu \frac{H_T^2}{2}] + \frac{1}{2} m [(\underline{v}_T - \sqrt{\mu \tau} \quad \underline{H}_T)]^2 = 0 , \tag{26}$$

where H_n has been replaced in accordance with (24). The second term is zero by (13) and the third term is zero by (25). Thus

$$[E] = 0 , \tag{27}$$

i.e., the internal energy is continuous. Now if E and τ are given, then every other thermodynamic quantity is determined. Therefore, since $[\tau] = 0 = [E]$, all the other thermodynamic quantities, including pressure, are continuous. It then follows from (13) that $\underline{H}_T^2$ is continuous. The fact that both the tangential and normal components of the magnetic field are continuous in magnitude means that the magnetic field is turned through an angle about the normal. Such a shock front is called a rotational discontinuity or a transverse shock.

An interesting feature of these waves is that, by a suitable choice of the coordinate system, we can always ensure that the fluid velocity is parallel to the field on each side of them. To achieve this, we use a coordinate system moving with velocity

$$\underline{v}_{1T} - \sqrt{\mu\tau}\ \underline{H}_{1T} = \underline{v}_{1T} - \sqrt{\mu\tau}\ \underline{H}_{2T} .$$

In the new system the ratio of each component of v to the corresponding component of H on either side of the discontinuity is $\sqrt{\mu\tau}$

$$\underline{v} = \sqrt{\mu\tau}\ \underline{H}_{1T} , \quad \underline{v}_2 = \sqrt{\mu\tau}\ \underline{H}_{2T} . \tag{28}$$

Thus in this system of coordinates the velocity is rotated with the magnetic field, its magnitude and angle to the normal remaining unchanged.

Stability of the tangential and rotational shocks in an incompressible fluid is discussed in the Section 8.

Case 3: $m \neq 0$, $[\tau] \neq 0$; fast and slow shocks: Such waves correspond to the shock waves of gas dynamics. They are characterized by a discontinuity of density and by the fact that the gas moves through them. From (14) and (15) we observe that, when $H_n \neq 0$, the jumps $[\underline{H}_T]$ and $[\tau \underline{H}_T]$ are parallel to the same vector $[\underline{v}_T]$, and therefore to each other. Hence $\underline{H}_{1T}$ and $\underline{H}_{2T}$ are parallel, i.e., the vectors $\underline{H}_1$, $\underline{H}_2$ and the normal to the surface are coplanar. This remains true even when $H_n = 0$ because then the relation (15) becomes $\tau_1 \underline{H}_{T1} = \tau_2 \underline{H}_{T2}$. The jump $[\underline{v}_T]$ lies in the same plane as $\underline{H}_1$ and $\underline{H}_2$. We can therefore assume, without loss of generality, that the vectors $\underline{v}_1$, and $\underline{v}_2$ themselves lie in this plane, so that the motion in the shock wave is two-dimensional.

From the Friedrichs-Kranzer equation (21) we observe that the flux m of the fast or slow shock ($m \neq 0$) satisfies the equation:

$$m^2 \ (\tau\, m^2 - \mu\, H^2) = - [\tau]^{-1} [p] \ (\tau\, m^2 - \mu\, H_n^2). \tag{29}$$

Thus, in analogy with the classification of sound waves we have

$$m_{slow}^2 \leq - [\tau]^{-1} [p] \leq m_{fast}^2 , \tag{30}$$

and

$$m_{slow}^2 \leq \mu\, H_n^2 \ / \ \tilde{\tau} \leq m_{fast}^2 . \tag{31}$$

Furthermore the jumps $[\underline{v}]$, $[\underline{H}]$, and $[\underline{\tau}]$ are given as

$$[\underline{v}] = k\, m\, (\mu\, H_n\, \underline{H} - \tilde{\tau}\, m^2\, \underline{n}), \tag{32}$$

$$[\underline{H}] = k\ m^2\ (\tilde{\underline{H}} - H_n\, \underline{n}), \tag{33}$$

$$[\tau] = -k\ (\tilde{\tau}\, m^2 - \mu\, H_n^2), \tag{34}$$

with an arbitrary k. From these relations we can easily calculate the jump in the magnitude of H, i.e.,

$$\begin{aligned} [H^2] &= 2\, \tilde{H} \cdot [\underline{H}] = k\, \tilde{m}^2\ (\tilde{H}^2 - H_n^2) \\ &= -m^2\ [\tau]\ \ (\tilde{\tau}\, m^2 - \mu\, H_n^2)^{-1}\ (\tilde{H}^2 - H_n^2). \end{aligned} \tag{35}$$

Gas dynamical shocks involve a rise in pressure and density, since the entropy increases across a shock. For hydromagnetic shocks we may state similarly:: If the entropy increases across a shock front, pressure and density also increase. (Note that, as we have seen in the earlier cases, there are shocks which do not involve jumps in entropy, density and pressure at all.) The proof of this statement follows from the identity

$$[E + \tilde{p}\tau] = - [\tau] \mu [H_T]^2 / 4, \tag{36}$$

where we have replaced $\underline{H}_T$ by H_T since $\underline{H}_{T1}$ and $\underline{H}_{T2}$ are parallel. This is the equation of the shock adiabatic in magnetized gas-dynamics. It differs from the ordinary equation by the presence of the right-hand side. To derive this identity we eliminate $[\underline{v}_T]$ from the equations (14) and (15) obtaining

$$m^2 [\tau H_T] = H_n^2 \mu [H_T] . \tag{37}$$

In order to eliminate $[\underline{v}_T]$ from equation (12), we write that equation as

$$[E + \frac{p}{\rho}] + \frac{1}{2} m^2 [\tau^2] + \frac{1}{2} [(\underline{v}_T - \frac{\mu H_n}{m} H_T)^2] + [\tau \mu H_T^2] - \frac{H_n^2}{4m^2} [\mu^2 H_T^2] = 0. \tag{38}$$

The third term is zero by equation (14) and so does not appear. In the last term we substitute m^2 from (37) and in the second term from (13), i.e.,

$$m^2 = \left\{ p_2 - p_1 + \frac{\mu}{2} (H_{T2} - H_{T1})^2 \right\} / (\tau_1 - \tau_2). \tag{39}$$

A simple calculation then yields the identity (36). From this relation one can also derive the fact that the increase of entropy across a shock is of the third order in ρ and H.

Since we have assumed the normal $\underline{n}$ to point in the direction in which the fluid crosses the shock front, the statement made preceeding the identity (36) implies that $[\tau] < 0$. In view of (31) and (35) it

follows that the magnetic field strength [H] rises across a fast shock and drops across a slow shock.

A wave front is called a "complete switch-on" shock if a tangential component of the magnetic field is produced through the wave (if it was absent ahead of the wave). It is called a "complete switch-off" shock if the tangential component of the magnetic field present before the wave is wiped out by it. Clearly, a complete switch-on shock must be fast or a complete switch-off shock must be slow. It can be shown that complete switch-on shocks exist only if the Alfven speed is supersonic ahead of the shock; and even then only if the shock strength lies below a critical value. If one lets the strength increase, the gain in tangential component of $\underline{H}$ first increases then decreases, becoming zero when the critical strength is reached. On the other hand, complete switch-off shocks always exist only if the Alfven speed behind the shock front is subsonic; if this speed is supersonic there, they exist only if the shock strength exceeds a critical value. It is, however, not possible to prescribe arbitrarily the tangential component of the magnetic field behind the shock front, and where it is possible to prescribe it at all, the shock may not be uniquely determined.

Gas-dynamical shocks have the property that the normal component of the flow velocity relative to the shock velocity $v_n - G$, is supersonic upstream of the shock wave and subsonic downstream. In the present case the normal flow velocity relative to the fast shock is greater than the fast sound speed ahead of the front and less than the fast sound speed but greater than (or equal to) the transverse sound speed behind the shock.

$$v_n - G > U_f, \qquad \text{on side 1, upstream}$$
$$a \leq v_n - G < U_f, \qquad \text{on side 2, downstream} \tag{40a}$$

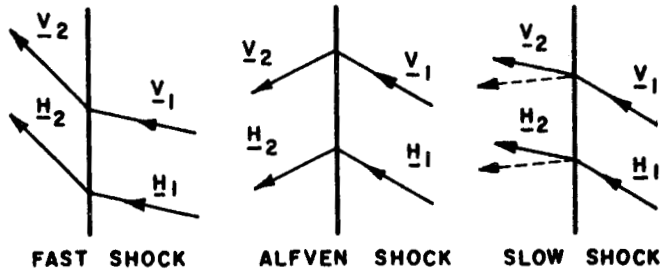
The equality sign holds only for the complete switching-on shocks. The normal flow velocity relative to a slow shock is greater than the slow sound speed ahead of the shock, but less than (or equal to) the transverse sound speed on both sides of the front.

$$v_n - H \leq a \qquad \text{on side 2, downstream}$$
$$a \geq v_n - G > U_s \qquad \text{on side 1, upstream} \tag{40b}$$

3. Transition of flow and field lines across a stationary plane shock

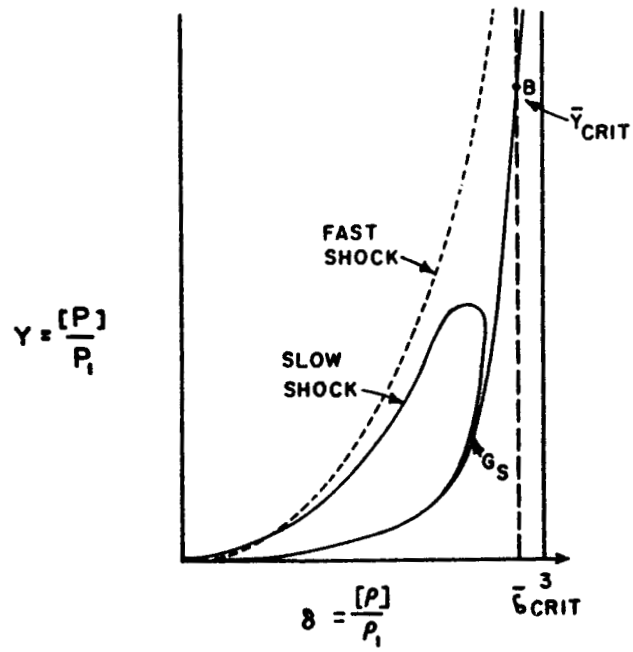
We have observed in previous sections that a frame of references can be so chosen that the streamlines are parallel to the lines of force. Figure 1 illustrates the transition of the lines of force and flow through stationary, plane hydromagnetic shocks. In fast shocks these lines are bent in a clockwise direction on crossing the shock from upstream to downstream, in slow shocks the bending is in the counterclockwise direction. The dashed arrows are used to indicate that the transverse magnetic field may change sign in slow shocks. In transverse shocks the lines of force and flow are just rotated through an angle (figure 1).

Another remark, which is worth making, follows from the shock adiabatic relation (36): The excess pressure ratio of a hydromagnetic shock corresponding to a given value of shock strength is, owing to the presence of transverse magnetic fields, greater than the excess pressure ratio in a pure gas-dynamical shock of the same



IT ILLUSTRATES THE TRANSITION THROUGH STATIONARY SHOCKS

FIGURE 1



ILLUSTRATES THE RELATIONSHIP BETWEEN $Y = [p]/p_1$ AND $X = \delta$, THE SHOCK STRENGTH G_S STANDS FOR GAS DYNAMIC SHOCK.

FIGURE 2

strength (figure 2).

4. Differential effects of shock waves

In order to study the spin and the distortion of a fluid element carried by a shock, it is essential that we evaluate the gradients of the hydromagnetic variables. The formal complexities encountered in deriving the results in full generality render the details of such an analysis forbidding. We shall therefore not make an attempt to that effect. There is, however, an interesting three-dimensional relation connecting the normal components of vorticity and electric current density and we shall reproduce that here.

As pointed out above we assume the shock surface $\Sigma(\underline{x}, t)$ is represented by a continuously differentiable function $x^i = x^i(u^1, u^2, t)$ where the x^i are rectangular Cartesian coordinates in an inertial frame and where u^α are the coordinates on $\Sigma(\underline{x}, t)$. The range of Latin indices, referring to surface tensors, is 1, 2. (see the appendix). The unit normal n^i, assumed continuously differentiable, is directed downstream. The relation (14), for steady flow, can be written as

$$\rho v_{1n} [v_\alpha] = \mu H_{1n} [H_\alpha] , \tag{41}$$

$$v_\alpha = v_i \frac{\partial x^i}{\partial u^\alpha} = v_i x^i_\alpha , \quad H_\alpha = H_i x^i_\alpha .$$

Let K_1 and K_2 be the principal curvatures of the shock surface. When the lines of curvature are taken as the Gaussian coordinates, Weingarten formulas of differential geometry gives

$$n^i_\alpha = K_\alpha x^i_\alpha \qquad , \ \alpha \text{ unsummed.} \tag{42}$$

If we differentiate (41) along the lines of curvature of the shock and use the relation (42), we readily get

$$\rho_1 \, v_{1n} \, [v_{\alpha,\beta}] = \mu \, H_{1n} \, [H_{\alpha,\beta}] + P_{\alpha\beta} \tag{43}$$

where

$$P_{\alpha\beta} = K_\beta \left\{ \rho_1 \, v_{1\beta} \, [v_\alpha] - \mu \, H_{1\beta} \, [H_\alpha] \right\} ,$$

β unsummed.

Multiply both sides of (43) by $\varepsilon^{\alpha\beta}$, where

$$\varepsilon^{\alpha\beta} = \frac{1}{\sqrt{g}} \, e^{\alpha\beta} , \; g = g_{\alpha\beta} , \; e^{11} = e^{22} = 0; \; e^{12} = -e^{21} = 1 , \tag{44}$$

$g_{\alpha\beta}$ x^i_α x^i_β is the surface metric (see the appendix).

The final result is

$$\rho_1 \, v_{1n} \, [w_n] = \mu \, H_{1n} \, [J_n] + \varepsilon^{\alpha\beta} \, P_{\alpha\beta} \tag{45}$$

where, as in the previous chapter, the quantities w_i and J_i denote the components of the vorticity and current density vectors. From (45) we observe that, unlike in gas dynamics, the shock waves are generally not vortex sheets. If $K_1 = K_2 = 0$, (e.g. a plane shock), the second term on the right-hand side of (45) disappears and we have a result similar to the one obtained in the last chapter.

5. Stability of the tangential and rotational shocks in an incompressible fluid

(a) Tangential shocks

Let the constants $\underline{v}_o$, p_o, and $\underline{H}_o$ be the value of the unperturbed flow and field quantities on each side of the discontinuity; and let $\underline{v}'$ and p'

and H' be the small perturbations. Furthermore set $\underline{u} = \sqrt{\frac{\mu}{\rho}}\ \underline{H}$. When we write

$$\underline{v} = \underline{v}_o + \underline{v}', \quad \underline{p} = \underline{p}_o + \underline{p}', \quad \underline{H} = \underline{H}_o + \underline{H}' \tag{46}$$

the equations of motion become:

$$\text{div}\ \underline{u}' = 0, \qquad \text{div}\ \underline{v}' = 0, \tag{47}$$

$$\frac{\partial \underline{U}'}{\partial t} = (\underline{u} \cdot \text{grad})\ \underline{v}' - (\underline{v} \cdot \text{grad})\ \underline{u}', \tag{48}$$

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \text{grad})\ \underline{v}' = -\frac{1}{\rho}\ \text{grad}\ p' - \underline{u} \times \text{curl}\ \underline{u}'$$

$$= -\frac{1}{\rho}\ \text{grad}\ (p' + \rho\ \underline{u} \cdot \underline{u}') + \underline{u} \cdot \text{grad}\ u'. \tag{49}$$

where, for notational simplicity, we have omitted the suffix 0. Taking the divergence of equation (49) and using (47) we obtain

$$\nabla^2\ (p' + \rho\ \underline{u} \cdot \underline{u}') = 0. \tag{50}$$

Let us assume that the wave front is a plane which we take to be $x = 0$; to which $\underline{v}$ and $\underline{u}$ are parallel. In each of the half spaces $x > 0$, $x < 0$ we seek all quantities $\underline{v}'$, $\underline{u}'$, and p' in a form proportional to $\exp\left\{ i\ (\underline{k} \cdot \underline{r} - \omega t \right\}$, where $\underline{k}$ is a vector in the y-z plane. Thus

$$\underline{F} = \underline{F}\ e^{\left\{ i\ (\underline{k} \cdot \underline{r} - \omega t) + \kappa\ x \right\}},$$

$$\frac{\partial \underline{F}}{\partial t} = -\ i\ \omega \underline{F},$$

$$\text{grad}\ F = (i\ \underline{k} + \underline{I}\ \kappa\ )\ \underline{F},$$

$$\text{div}\ F = (i\ \underline{k} + \underline{I}\ \kappa\ ) \cdot \underline{F},$$

$$\text{curl}\ F = (i\ \underline{k} + \underline{I}\ \kappa\ ) \times \underline{F}\ , \tag{51}$$

where $\underline{I}$ is the unit vector in the x-direction.

Thus the equation (50) reduces to

$$(\kappa^2 - k^2)\,(p' + \rho\,\underline{u}\cdot\underline{u}') = 0, \tag{52}$$

so that $\kappa = \pm k$. Since we want κ to be the damping factor, we must put $\kappa = k$ for $x < 0$, and $\kappa = -k$ for $x > 0$. Now from (51) and (48-49) we have

$$-i\omega\,\underline{u}' = \left\{\underline{u}\cdot(i\,\underline{k} + \underline{I}\,\kappa)\right\}\underline{v}' - \left\{\underline{v}\cdot(i\,\underline{k} + \underline{I}\,\kappa)\right\}\underline{u}', \tag{53}$$

$$-i\,\omega\,\underline{v}' + \left\{\underline{v}\cdot(i\,\underline{k} + \underline{I}\,\kappa)\right\}\underline{v} = -\frac{1}{\rho}\left\{(i\,\underline{k} + \underline{I}\,\kappa)\,p' + \rho\underline{u}\cdot(i\,\underline{k} + \underline{I}\,\kappa)\,u'\right\} +$$

$$+\ \underline{u}\cdot\left\{(i\,\underline{k} + \underline{I}\,\lambda)\right\}\underline{u}' \tag{54}$$

The x-component of these equations are (note that $\underline{u}_x = 0$, $\underline{v}_x = 0$):

$$-i\,\omega\,u_x' = i\,(\underline{u}\cdot\underline{k})\,v_x' - i\,(\underline{v}\cdot\underline{k})\,u_x', \tag{55}$$

$$-i\,\omega\,v_x' = -\frac{1}{\rho}\,\kappa\,(p' + \rho\,\underline{u}\cdot\underline{u}') + i\,(\underline{u}\cdot\underline{k})\,u_x'. \tag{56}$$

When we eliminate v_x' from these equations we obtain

$$p' + \rho\,\underline{u}\cdot\underline{u}' = -u_x'\,\frac{i\,\rho}{(\underline{k}\cdot\underline{u})\kappa}\left\{(\omega - \underline{v}\cdot\underline{k})^2 - (\underline{k}\cdot\underline{u})\right\} \tag{57}$$

Now let $\zeta = \zeta(y,z,t)$ be the displacement of the surface of discontinuity in the x-direction due to the perturbation. According to the discussion of the tangential shocks in Section 2, the following conditions must be satisfied on both sides of this displaced surface:

$$m = 0,\quad H_n = 0,\quad p + \left[\frac{\mu\,H_T^2}{2}\right] = 0. \tag{58}$$

Therefore:

$$p + \rho (\underline{u} + \underline{u}')^2 \cong p' + \rho \underline{u} \cdot \underline{u}' = 0,$$

$$u_{n1} + u'_{n1} \cong u'_{x1} - \underline{u}_1 \cdot \text{grad } \zeta = 0,$$

$$u_{n2} + u'_{n2} \cong u'_{x2} - \underline{u}_2 \cdot \text{grad } \zeta = 0. \qquad (59)$$

The relation m = 0, is automatically satisfied. To study the stability we set

$$\gamma = \text{constant x exp} \left\{ i (k \cdot r - \omega t) \right\} \qquad (60)$$

then

$$\text{grad } \gamma = i k \gamma. \qquad (61)$$

Now from (58) and (59), we have:

$$-u'_{x2} - \frac{i \rho}{(\underline{k} \cdot \underline{u}_2)\kappa} \left\{ (\omega - \underline{k} \cdot \underline{v}_2)^2 - (\underline{k} \cdot \underline{u}_2)^2 \right\} +$$

$$+ u'_{x1} \frac{i \rho}{(\underline{k} \cdot \underline{u}_1)\kappa} \left\{ (\omega - \underline{k} \cdot \underline{v}_1)^2 - (\underline{k} \cdot \underline{u}_1)^2 \right\} = 0 \qquad (62)$$

But $(59)_{2,3}$ and (61) give

$$u'_{x1} = \underline{u}_1 - i \underline{k} \gamma = i \gamma \underline{u}_1 \cdot \underline{k}; \quad u'_{x2} = i \gamma \underline{u}_2 \cdot \underline{k}. \qquad (63)$$

From (62) and (63) we finally derive an equation giving the possible values of ω:

$$(\omega - \underline{k} \cdot \underline{v}_1)^2 + (\omega - \underline{k} \cdot \underline{v}_2)^2 = (\underline{k} \cdot \underline{u}_1)^2 + (\underline{k} \cdot u_2)^2. \qquad (64)$$

This equation has no complex roots if

$$2 (\underline{k} \cdot \underline{u}_1)^2 + 2 (\underline{k} \cdot \underline{u}_2)^2 - \left\{ \underline{k} \cdot [\underline{v}] \right\}^2 > 0 \qquad (65)$$

or

$$\left\{ 2 u_{1i} u_{1j} + 2 u_{2i} u_{2j} - (v_{2i} - v_{2j})(v_{2j} - v_{1j}) \right\} k_i k_j > 0 \qquad (66)$$

This quadratic form is positive definite if the trace and determinant of the tensor of rank two in the braces are both positive. Let us write this tensor in matrix form (note $\underline{u}$ and $\underline{v}$ have no x-components):

$$\begin{Vmatrix} 2u_{22}^2 + 2u_{12}^2 - [v_2]^2 & 2u_{22}u_{23} + 2u_{12}u_{13} - [v_2][v_3] \\ 2u_{23}u_{22} + 2u_{13}u_{12} - [v_2][v_3] & 2u_{23}^2 + 2u_{13}^2 - [v_3]^2 \end{Vmatrix} \quad (67)$$

The trace is

$$2u_{22}^2 + 2u_{12}^2 + 2u_{23}^2 + 2u_{13}^2 - [v_2]^2 - [v_3]^2 > 0,$$

or

$$(u_1^2 + u_2^2) > [v]^2 \quad ,$$

or

$$H_1^2 + H_2^2 > \frac{\rho}{2\mu} [\underline{v}]^2 \quad . \quad (68)$$

The determinant can be similarly calculated and takes the following form

$$(\underline{H}_1 \times \underline{H}_2)^2 \geq \frac{\rho}{2\mu} \left\{ (\underline{H}_1 \times \underline{v})^2 + (\underline{H}_2 \times \underline{v})^2 \right\} \quad . \quad (69)$$

In the ordinary fluid dynamics, tangential discontinuities are always unstable with respect to infinitesimal perturbations, and so are rapidly broadened into turbulent regions. From the above analysis it appears that a magnetic field has a stabilizing effect on the motion of a conducting fluid. The reason is that a perturbation involving fluid displacements transverse to the field leads to a stretching of the lines of magnetic force frozen in it, and therefore gives rise to forces which tend to restore the unperturbed flow.

When we take into account the existence of a small but finite

viscosity and electric resistence in the fluid, it emerges that such tangential discontinuities cannot exist indefinitely, even if the conditions (68) and (69) are fulfilled. Although no turbulence occurs, the sharp discontinuity is replaced by a gradually widening transitional region. In this region the velocity and the magnetic field change smoothly from one value to another. To prove this we take the x-axis in the direction of the normal to the discontinuity. Assuming all quantities to depend on x and t, we can write the transverse components of the equations (17) and (21) of Chapter 1 for the incompressible fluid as

$$\frac{\partial \underline{H}_T}{\partial t} = \eta \frac{\partial^2 \underline{H}_T}{\partial x^2}, \qquad \frac{\partial \underline{v}_T}{\partial t} = \frac{\partial^2 \underline{v}_T}{\partial x^2} \tag{70}$$

These equations resemble the heat conduction equations. Thus, as in the theory of thermal conduction, the discontinuities in $\underline{v}$ and $\underline{H}$ are gradually smoothed into a transitional region, whose width increases as the square root of time. Since the coefficients in the above two equations are different, the widths δ_v and δ_H of the transitional region for the velocity and the magnetic field are also different

$$\delta_v \sim \sqrt{\nu t}, \qquad \delta_H \sim \sqrt{\eta t}. \tag{71}$$

(b) Rotational shocks

If we follow the procedure of the case above in connection with the stability of the rotational wave fronts, it emerges that these waves, in an incompressible fluid, are stable with respect to infinitesimal

perturbations, whatever the strength of the magnetic field. This becomes possible due to the additional relation (28). We shall, however, prove that like tangential discontinuities, they cannot have constant widths, but are gradually smoothed out by the dissipative effects of viscosity and electric resistance of the fluid. To prove this result we assume all quantities to depend only on the coordinate x and time t. The relations div $\underline{v} = 0 =$ div $\underline{H}$, lead to:

$$v_x = \text{constant}; \quad H_x = \text{constant} \quad . \tag{72}$$

Moreover we use the coordinate system in which $\underline{v}$ and $\underline{H}$ are related by the equations (28). Thus from (28) and (72) we have

$$v_x = u_x \ , \tag{73}$$

where u is defined, as in the first case above, by the relation $\underline{u} = \sqrt{\mu \tau}\ \underline{H}$. The general equations (19) and (20) of Part 1 reduce to

$$\frac{\partial \underline{u}_T}{\partial t} + u_x \frac{\partial \underline{u}_T}{\partial x} = u_x \frac{\partial \underline{v}_T}{\partial x} + \eta \frac{\partial^2 \underline{u}_T}{\partial x^2} \tag{74}$$

and

$$\frac{\partial \underline{v}_T}{\partial t} + u_x \frac{\partial \underline{v}_T}{\partial x} = u_x \frac{\partial \underline{u}_T}{\partial x} + \frac{\partial^2 \underline{v}_T}{\partial x^2} \tag{75}$$

wherein we have used the relation (56). Now because of the relation (28), the difference $\underline{v}_T - \underline{u}_T$ tends to zero for $x \Rightarrow \pm \infty$. Therefore, this difference must be small in the transitional layer in comparison with the sum $\underline{v}_T + \underline{u}_T$. Adding the equation (73) and (74), and neglecting $\underline{v}_T - \underline{u}_T$, we obtain

$$\frac{\partial}{\partial t} \left(\underline{v}_T + \underline{u}_T\right) = \frac{1}{2} \left(\nu + \eta\right) \frac{\partial^2}{\partial x^2} \left(\underline{v}_T + \underline{u}_T\right). \tag{76}$$

From this we see that the width δ of the discontinuity varies in a manner given by

$$\delta \sim \sqrt{\left(\nu + \eta\right) t} \ . \tag{77}$$

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Appendix

Geometry of Surfaces in Space

Let (x^1, x^2, x^3) be the Cartesian coordinates of a point in Euclidean space R. A surface is defined in general as the locus of a point whose coordinates are functions of two independent parameters. Thus the equations of a surface are of the form

$$x^i = \phi^i (u^1, u^2), \tag{1}$$

where ϕ^i are differentiable functions of the two parameters u^1, u^2 such that the functional matrix $\left\| \partial\varphi^i / \partial u^\alpha \right\|$, $\alpha = 1, 2$; has rank 2. These functions are supposed to have all the smoothness needed in the following analysis. One commonly refers to the parameters u^1, u^2 as the curvilinear coordinates of the surface. Any point on the surface is uniquely determined by two numbers u^1, u^2. It will be assumed that Greek indices have range 1, 2 and Latin indices the range 1, 2, 3.

Consider the quantities x^i_α defines by

$$x^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}, \quad (i = 1, 2, 3; \alpha = 1, 2). \tag{2}$$

These quantities are the components of two contravariant vectors in the space R and three covariant vectors on the surface S. Now any vector tangent to a curve on the surface S is said to be tangent to this surface Thus the two space vectors having the components x^i_1 and x^i_2, and being tangent to the u^1 and u^2 coordinate curves, are tangent to the surface S. Moreover

$$dx^i = \frac{\partial x^i}{\partial u^\alpha} du^\alpha = x^i_\alpha du^\alpha, \tag{3}$$

in which the quantities du^{α} and dx^{i} are called, respectively, the surface and space components of the same vector, e.g., a displacement on the surface S, and we have used the summation convention.

Since the matrix $\left\| x^{i}_{\alpha} \right\|$ has rank 2 it follows from the implicit function theorem that two of the three equations (1) can be solved for the variables u^1, u^2. The solution functions can then be used to eliminate the variables u^1, u^2 from the remaining equation (1). Hence the surface S can always be represented locally by an equation of the form

$$f(x^1, x^2, x^3) = 0 \quad . \tag{4}$$

When we eliminate the coordinate x^i from (4) by means of (1), the resulting equations must be satisfied identically in the independent variables u^1 and u^2. Thus

$$\frac{\partial f}{\partial x^i} \quad x^{i}_{\alpha} = 0. \tag{5}$$

Since x^{i}_{α} are tangent vectors to the surface, we observe from (5) that the gradient of the function is normal to the surface S. The unit normal vector is therefore given by

$$n_i = \frac{\partial f / \partial x^i}{\left| \text{grad } f \right|} \tag{6}$$

In the space R we have

$$ds^2 = \delta_{ij} \, dx^i \, dx^j \quad . \tag{7}$$

If ds lies on the surface, then by using (3), the relation (7) becomes

$$ds^2 = g_{\alpha\beta} \, dx^{\alpha} \, dx^{\beta} \quad , \tag{8}$$

where we have put

$$g_{\alpha\beta} = \frac{\partial y^i}{\partial u^\alpha} \quad \frac{\partial y^i}{\partial u^\beta} \quad . \tag{9}$$

The quadratic differential form (7) is called the first fundamental form of the surface S.

We now have defined all the quantities occurring in the identity

$$g^{\alpha\beta} \quad x^i_\beta \quad x^j_\beta \quad = \quad \delta^{ij} \; - \; n_i \, n_j \quad , \tag{10}$$

used in Chapter 4, equation (12).

When we are dealing with wave fronts $\Sigma(t)$, then the equation for a surface is given as

$$f(\underline{x}, \; t) \; = \; 0 \tag{11}$$

Differentiating both sides of (11) with respect to t, we have

$$\frac{\partial f}{\partial t} \quad + \quad \text{grad } f \; \cdot \; \frac{d\underline{x}}{dt} \quad = \quad 0, \tag{12}$$

When the normal speed of advance of the wave is G, there results

$$\frac{d\underline{x}}{dt} \quad = \quad G \; \underline{n} \quad . \tag{13}$$

From (12), (13) and (6) we readily obtain

$$G \; = \; - \; \frac{\partial f}{\partial t} \Bigg/ \left| \; \text{grad } f \; \right| \quad . \tag{14}$$